Further Analytical Results on the Joint Detection of Cochannel Signals Using Diversity Arrays

Stephen J. Grant and James K. Cavers

***Abstract*—We derive a fully analytical expression for the union bound on symbol-error rate for the joint detection of several cochannel fading signals using a diversity antenna array, putting numerical results previously published on a firm analytical base. We prove that with pilot-based multiuser channel estimation, any number of users can enjoy diversity order equal to the number of antennas, and we quantify the performance penalty relative to single-user binary phase-shift keying as a function of number of users, constellation density, and number of antennas. We also demonstrate the interdependence of all users participating in the detection process.**

***Index Terms*—Antenna arrays, cochannel interference, fading channels, multidimensional signal detection, multiuser channels.**

I. INTRODUCTION

IN [1], we considered reception of the cochannel signals from M mobile users at a base station with an array of L diversity antennas. We demonstrated that maximum-likelihood joint detection of all user signals can provide asymptotic diversity order L, regardless of the number of users. This is in strong contrast to nulling and minimum mean-square error (MMSE) combining, in which each additional user reduces the order of diversity by one for all users, and the maximum number of users is limited by the number of antennas. However, our expression for pairwise-error probability was left in terms of a pair of eigenvalues that had to be evaluated numerically. In this letter, we derive a closed-form analytical expression for the ratio of the two nonzero eigenvalues, which results in a fully analytical solution for the pairwise-error probability, and hence the union bound on symbol-error rate (SER).

Having this analytical solution allows the derivation of several new results. First, some parameters of the joint detection process are shown not to affect the resulting SER. Second, we prove that all users enjoy L-fold asymptotic diversity, in contrast to [1], where it was simply observed from numerical calculation. Third, we obtain expressions for the constant signal-to-noise ratio (SNR) loss per additional user as a function of the phase-shift keying (PSK) constellation size and the number of antennas.

II. ANALYTICAL SOLUTION FOR SER

A. Summary of the Method

In [1], the pairwise-error probability is defined as the probability that the maximum-likelihood detector chooses the erroneous data vector $\mathbf{c}_i = (c_{i1}, c_{i2}, \ldots, c_{iM})$ instead of the transmitted data vector $\mathbf{c}_j = (c_{j1}, c_{j2}, \ldots, c_{jM})$, where the data symbols c_{im} and c_{jm} are for the mth user. The data symbols are drawn from a PSK constellation of size Q and unit radius. The resulting pairwise-error probability is

$$P_{2_{ij}} = \frac{1}{(1-r_{ij})^{2L-1}} \sum_{k=0}^{L-1} \binom{2L-1}{k} (-r_{ij})^k \qquad (1)$$

where $r_{ij} = \lambda_{ij1}/\lambda_{ij2}$ is the ratio of the two nonzero eigenvalues of the matrix $\mathbf{RF}_{ij}$ defined below. Our convention is that λ_{ij1} is positive and λ_{ij2} is negative; consequently, r_{ij} is negative.

The union bound on SER for user m is then

$$P_{s_{mj}} = \sum_{i \in C_{mj}} P_{2_{ij}} \qquad (2)$$

where C_{mj} is the set of signal vectors that differ in their mth position from $\mathbf{c}_j$. In [1], the eigenvalue ratio r_{ij} is calculated numerically in order to evaluate the performance of joint detection. In Section II-B, we provide a closed-form analytical solution.

The $(M+1) \times (M+1)$ matrix $\mathbf{RF}_{ij}$ (see [1, eq. (21)]) has factors as in (3), shown at the bottom of the next page, and $\mathbf{F}_{ij} = \mathbf{F} + \mathbf{F}^{\dagger}$ with (4), shown at the bottom of the next page, where $*$ and $\dagger$ represent complex conjugate and complex conjugate transpose, respectively. In (3) and (4), A_m is related to the transmit power P_m of the mth user's signal by $A_m = \sqrt{2P_m}$; $\beta_m = \rho_m \sigma_{g_m}/\sigma_{v_m}$, where ρ_m is the complex correlation coefficient between the true channel gain g_{lm} from user m to antenna l and its estimate v_{lm}, and σ_{g_m} and σ_{v_m} are the standard deviations of g_{lm} and v_{lm}, respectively. Note that for perfect channel state information (CSI), we have $v_{lm} = g_{lm}$, $\rho_m = 1$, and $\beta_m = 1$.

For simplicity in obtaining asymptotic results, in (3), we have written the SNR $A_m^2 \sigma_{g_m}^2 / N_o$ of user m as $K_m \Gamma$, where N_o is the power spectral density of the additive white Gaussian noise (AWGN) process, K_m is a scale factor, and Γ is a reference SNR. The reference SNR is arbitrary, but logical choices might be the arithmetic or geometric average of the M users' SNRs, or one user's SNR in particular—perhaps that of the strongest user. In terms of the notation in [1], $\Gamma_m = K_m \Gamma$. Although (3) and (4) are the same quantities as their counterparts in [1, eqs. (20), (17)], they are written in a slightly different form (the former by use of the relation $\sigma_{v_m}^2 = |\rho_m|^2 \sigma_{g_m}^2 / |\beta_m|^2$ and the reference SNR) for convenience in eigenvalue derivation.

Paper approved by K. B. Letaief, the Editor for Wireless Systems of the IEEE Communications Society. Manuscript received April 10, 1999; revised November 22, 1999 and March 30, 2000.

The authors are with the School of Engineering Science, Simon Fraser University, Burnaby, BC V5A 1S6, Canada (e-mail: grantq@sfu.ca; cavers@sfu.ca).

Publisher Item Identifier S 0090-6778(00)09889-5.

B. Derivation of Eigenvalue Ratio

Since the rank of the matrix $\mathbf{RF}_{ij}$ is only two, the characteristic polynomial of the matrix may be expressed in the following form:

$$\det[\mathbf{RF}_{ij} - \lambda\mathbf{I}] = \lambda^{M-1}(\lambda^2 + \alpha_1\lambda + \alpha_2) = 0. \quad (5)$$

According to Bôcher's formula [2], the coefficients of the quadratic polynomial are given by $\alpha_1 = -T_1$ and $\alpha_2 = -(1/2)(\alpha_1 T_1 + T_2)$, where $T_k = \mathrm{trace}[(\mathbf{RF}_{ij})^k]$. Knowing the coefficients allows us to calculate the ratio of the two nonzero eigenvalues as

$$r_i = \frac{T_1 + \sqrt{2T_2 - T_1^2}}{T_1 - \sqrt{2T_2 - T_1^2}}. \quad (6)$$

Note that the subscript j has been dropped since it is assumed, without loss of generality, that $\mathbf{c}_j$ is the all-ones vector. Evaluation of the various traces is made easier if the matrix $\mathbf{R}$ is written as the sum of two simpler matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, where $\mathbf{R}_1$ contains the first column and first row of $\mathbf{R}$ with the remainder of the elements set to zero, and $\mathbf{R}_2$ contains the main diagonal of $\mathbf{R}$ (excluding the top left element) with the remainder of the elements set to zero. T_1 and T_2 are then given by

$$\begin{aligned} T_1 &= \mathrm{trace}[(\mathbf{R}_1 + \mathbf{R}_2)(\mathbf{F} + \mathbf{F}^\dagger)] \\ T_2 &= \mathrm{trace}[(\mathbf{R}_1 + \mathbf{R}_2)(\mathbf{F} + \mathbf{F}^\dagger)(\mathbf{R}_1 + \mathbf{R}_2)(\mathbf{F} + \mathbf{F}^\dagger)]. \end{aligned} \quad (7)$$

The above expressions require evaluation of the traces of several matrix products: four twofold products in the former, and 16 fourfold products in the latter. Fortunately, the sparse nature and special form of the matrices make some of the traces zero and limit the complexity of others. We have not shown these intermediate calculations because of their size.

After laborious evaluation of the nonzero traces using $c_{jm} = 1$ for all m in (3) and (4), we find the ratio of eigenvalues to be

$$r_i = \frac{a_i\Gamma + \sqrt{b_i\Gamma^2 + 2a_i\Gamma}}{a_i\Gamma - \sqrt{b_i\Gamma^2 + 2a_i\Gamma}} \quad (8)$$

where

$$\begin{aligned} a_i &= \sum_{m=1}^{M} |\rho_m|^2 K_m (1 - \mathrm{Re}[c_{im}]) \\ b_i &= 2\sum_{n=1}^{M}\sum_{m=1}^{M} |\rho_m|^2 K_n K_m (1 - \mathrm{Re}[c_{im}]) \\ &\quad + \left(\sum_{m=1}^{M} |\rho_m|^2 K_m \mathrm{Re}[c_{im}]\right)^2 - \left(\sum_{m=1}^{M} |\rho_m|^2 K_m\right)^2. \end{aligned} \quad (9)$$

In the derivation, we have made explicit use of $\mathrm{Re}[c_{im}]^2 + \mathrm{Im}[c_{im}]^2 = 1$ for PSK. In (1), (8), and (9), we have an analytical expression for the pairwise-error probability, and therefore the union bound on detection SER (2).

We can draw two immediate inferences from this result: first, the pairwise-error probability depends only on the erroneous data vector $\mathbf{c}_i$ and each user's own SNR $K_m\Gamma$ and channel estimate correlation coefficient ρ_m. It does not depend separately on A_m or β_m, a fact that was not evident in the numerical approach of [1].

Second, in the special case of equipower users and perfect CSI, we obtain an interesting interpretation of SNR. With perfect CSI, we have all $\rho_m = 1$, and the parameter b_i in (9) simplifies to $b_i = a_i^2$. If the cochannel signals arrive at the base station with equal average powers, perhaps as a result of power control, then $K_m = 1$ for all users, and the parameter a_i in (9) simplifies to $a_i = \sum_{m=1}^{M}(1 - \mathrm{Re}[c_{im}])$. Consequently, from (8), we see

$$\mathbf{R} = \mathbf{N}_o \begin{bmatrix} 1 + \sum_{m=1}^{M} K_m\Gamma & \frac{|\rho_1|^2}{A_1\beta_1^*} c_{j1}K_1\Gamma & \frac{|\rho_2|^2}{A_2\beta_2^*} c_{j2}K_2\Gamma & \cdots & \frac{|\rho_M|^2}{A_M\beta_M^*} c_{jM}K_M\Gamma \\ \frac{|\rho_1|^2}{A_1\beta_1} c_{j1}^*K_1\Gamma & \frac{|\rho_1|^2}{A_1^2|\beta_1|^2} K_1\Gamma & 0 & \cdots & 0 \\ \frac{|\rho_2|^2}{A_2\beta_2} c_{j2}^*K_2\Gamma & 0 & \frac{|\rho_2|^2}{A_2^2|\beta_2|^2} K_2\Gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{|\rho_M|^2}{A_M\beta_M} c_{jM}^*K_M\Gamma & 0 & 0 & 0 & \frac{|\rho_M|^2}{A_M^2|\beta_M|^2} K_M\Gamma \end{bmatrix} \quad (3)$$

$$\mathbf{F} = \begin{bmatrix} 0 & A_1\beta_1(c_{j1} - c_{i1}) & A_2\beta_2(c_{j2} - c_{i2}) & \cdots & A_M\beta_M(c_{jM} - c_{iM}) \\ 0 & 0 & A_1A_2\beta_1^*\beta_2(c_{i1}^*c_{i2} - c_{j1}^*c_{j2}) & \cdots & A_1A_M\beta_1^*\beta_M(c_{i1}^*c_{iM} - c_{j1}^*c_{jM}) \\ 0 & 0 & 0 & \cdots & A_2A_M\beta_2^*\beta_M(c_{i2}^*c_{iM} - c_{j2}^*c_{jM}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (4)$$

that the eigenvalue ratio, and therefore the SER, depend only on an effective SNR

$$\Gamma_{i,\,\mathrm{eff}} = \Gamma \sum_{m=1}^{M} (1 - \mathrm{Re}[c_{im}]). \tag{10}$$

Further, if the modulation is binary phase-shift keying (BPSK), then a_i becomes twice the Hamming distance $d_{i,\,H}$ between $\mathbf{c}_j$ and $\mathbf{c}_i$ (the number of positions in which the transmitted and erroneous data vectors differ), since we have taken the transmitted vector $\mathbf{c}_j$ to be all ones. Thus, the SER depends only on the effective SNR $\Gamma_{i,\,\mathrm{eff}} = 2d_{i,\,H}\Gamma$, which is intuitively satisfying.

III. Asymptotic Behavior

Much insight into the behavior of joint detection may be gained by considering its asymptotic performance; that is, the performance as the common SNR Γ becomes large. We would hope to see the eigenvalue ratio (8) continue to increase with Γ, in order to drive the pairwise-error probability (1) toward zero. The questions are whether the ratio does increase without limit and, if so, how quickly.

A. Asymptotic-Error Rate

A realistic model of imperfect CSI allows the channel estimation correlation coefficients ρ_m to improve as the SNR increases, thereby eliminating the error floor (an irreducible SER for large SNR). As an example, we use the pilot-based multiuser channel estimation scheme of [3], where it is shown that, for large SNR, the normalized channel estimation error variance for user m, given by $1-|\rho_m|^2$, is inversely proportional to that user's own SNR $\Gamma_m = K_m\Gamma$. The constant of proportionality, denoted here as k_m, depends on the number of users M as well as the parameters of the channel estimation scheme such as interpolator order, frame length, Doppler fade rate, and the choice of training sequences. Typical values for k_m are on the order of 10^{-1}. We now have $|\rho_m|^2 = 1-(k_m/K_m)\Gamma^{-1}$. Substitution into (9) and (8) gives the eigenvalue ratio in the following alternative form after collecting terms with like powers of Γ:

$$r_i = \frac{\alpha_i\Gamma + \beta_i + \sqrt{\alpha_i^2\Gamma^2 + \gamma_i\Gamma + \delta_i}}{\alpha_i\Gamma + \beta_i - \sqrt{\alpha_i^2\Gamma^2 + \gamma_i\Gamma + \delta_i}} \tag{11}$$

where the coefficients are

$$\begin{aligned}
\alpha_i &= \sum_{m=1}^{M} K_m(1-\mathrm{Re}[c_{im}])\\
\beta_i &= -\sum_{m=1}^{M} k_m(1-\mathrm{Re}[c_{im}])\\
\gamma_i &= 2\left(1+\sum_{m=1}^{M} k_m\mathrm{Re}[c_{im}]\right)\sum_{m=1}^{M} K_m(1-\mathrm{Re}[c_{im}])\\
\delta_i &= \sum_{m=1}^{M} k_m\mathrm{Re}[c_{im}]\left(2+\sum_{m=1}^{M} k_m\mathrm{Re}[c_{im}]\right)\\
&\quad -\sum_{m=1}^{M} k_m\left(2+\sum_{m=1}^{M} k_m\right).
\end{aligned} \tag{12}$$

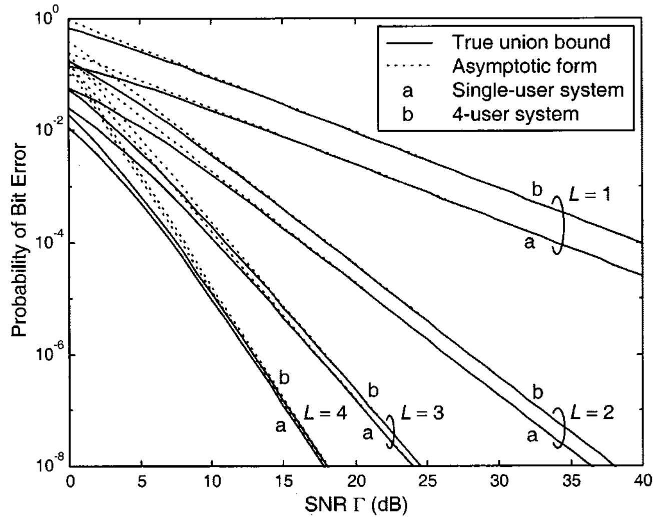

Fig. 1 Comparison of the true union bound on symbol-error rate and its asymptotic form for BPSK modulation, equipower users, and perfect CSI.

To obtain the asymptotic-error rate, we let the SNR Γ in (11) become very large. The eigenvalue ratio approaches

$$r_{i,\,\mathrm{asymp}} = \frac{4\alpha_i^2}{2\alpha_i\beta_i - \gamma_i}\Gamma = \frac{-2\sum_{m=1}^{M} K_m(1-\mathrm{Re}[c_{im}])}{1+\sum_{m=1}^{M} k_m}\Gamma \tag{13}$$

which increases without bound as Γ increases. As a check, the special case of equipower users, perfect CSI, and BPSK modulation produces $r_{i,\,\mathrm{asymp}} = -4d_{i,\,H}\Gamma$, which is identical to the result obtained by setting $a_i = 2d_{i,\,H}$ and $b_i = a_i^2$ in (8) and allowing Γ to approach infinity.

The pairwise-error probability is obtained by substituting (13) into (1) and noting that the asymptotic form of the result is determined by the last term in the summation, giving $P_{2_i,\,\mathrm{asymp}} = \binom{2L-1}{L-1}(-r_i)^{-L}$. Substitution of (13) yields, finally

$$P_{2_i,\,\mathrm{asymp}} = 2^{-L}\binom{2L-1}{L-1}\left(\frac{\sum_{m=1}^{M} K_m(1-\mathrm{Re}[c_{im}])}{1+\sum_{m=1}^{M} k_m}\right)^{-L}\Gamma^{-L}. \tag{14}$$

This analytical result shows clearly that the pairwise-error probability, and therefore the union bound on SER, varies asymptotically as Γ^{-L}, regardless of the number of users. That is, all users enjoy L-fold diversity with no error floor, a result that was apparent only from numerical results in [1] for the special case of equipower users.

This behavior is illustrated in Fig. 1, which compares the true union bound, obtained analytically using (8), with the asymptotic form, obtained from (14). Note that this graph considers equipower users (all $K_m = 1$) so that Γ is the common SNR.

Numerical comparison of the true union bound and the simulated SER can be found in [1]. Both the graph and (14) demonstrate that the effect of additional users is simply to shift the curves to the right by an SNR penalty that is constant, measured in decibels. Below, we use our asymptotic expressions to obtain several new results for the SNR penalty.

B. SNR Penalty for Imperfect Channel Estimation

We have expressed the quality of multiuser pilot-based channel estimation by the coefficients k_m. It is clear from (14) that imperfect estimation degrades the SNR by an asymptotic factor of $1+\sum_{m=1}^{M} k_m$ for every pairwise error event. This is one illustration of the interdependence of data detection for different mobiles, since any mobile with poor channel estimation (a large value of k_m) degrades the performance of all users, even those with perfect CSI ($k_m = 0$).

C. SNR Penalty for Additional Users

This performance degradation is quantified by the additional SNR required to support M users at a fixed symbol-error rate in the asymptotic region, compared with that required to support a single user at the same error rate. Here, we derive an expression for the SNR penalty experienced by a system with equipower users and perfect CSI ($K_m = 1$ and $k_m = 0$). The asymptotic SER is then

$$P_{s,\,\mathrm{asymp}} = 2^{-L}\binom{2L-1}{L-1}\sum_{i\in C_1}\left(\sum_{m=1}^{M}(1-\mathrm{Re}[c_{im}])\right)^{-L}\Gamma_M^{-L} \tag{15}$$

where the subscript j is dropped because the transmitted vector $\mathbf{c}_j$ is all ones, and subscript m is dropped because the mobiles have identical conditions, allowing us to consider only C_1, the set of vectors that differ from unity in the first position. The SNR has been denoted Γ_M to emphasize the number of users. For the reference single-user system with Q points in the PSK constellation, we have $i \in \{1, 2, \ldots, Q-1\}$ and $\mathrm{Re}[c_{i1}] = \cos[2\pi i/Q]$, resulting in the simple expression $2^{-L}\binom{2L-1}{L-1}\sum_{i=1}^{Q-1}(1-\cos[2\pi i/Q])^{-L}\Gamma_1^{-L}$ for asymptotic SER.

Define $U_M = \Gamma_M/\Gamma_1$ as the penalty factor by which the SNR must be increased to maintain the same SER as a single-user system. From (15), we have the SNR penalty as

$$U_M = \left[\frac{\sum_{i\in C_1}\left(\sum_{m=1}^{M}(1-\mathrm{Re}[c_{im}])\right)^{-L}}{\sum_{i=1}^{Q-1}(1-\cos[2\pi i/Q])^{-L}}\right]^{1/L}. \tag{16}$$

A plot of (16) in Fig. 2 for BPSK modulation shows that, for a given number of antennas L, there is a constant penalty in decibels for each additional user. Moreover, as the number of antennas increases, the degradation per additional user decreases dramatically. For example, for a single antenna ($L = 1$), the degradation is approximately 2 dB per additional user, whereas it is less than 0.1 dB per additional user for $L = 4$. Thus, not only do multiple antennas give better performance for a single user, they maintain that performance better than a single antenna in the face of an increasing user population.

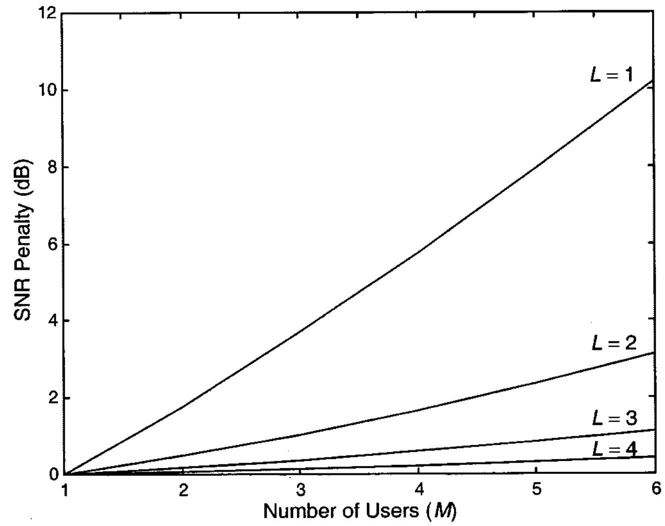

Fig. 2 SNR penalty in the asymptotic region for BPSK, equipower users, and perfect CSI.

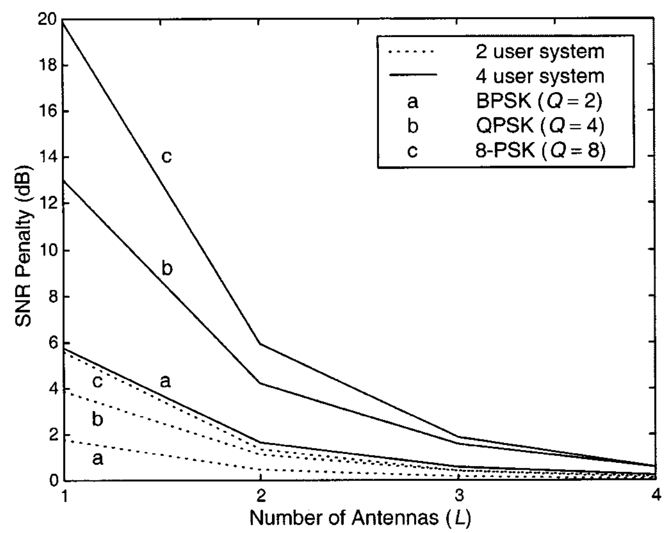

Fig. 3 SNR penalty in the asymptotic region for several different modulation formats with equipower users and perfect CSI.

D. SNR Penalty for Larger Constellations

Fig. 3 plots the SNR penalty for three different modulation formats, namely BPSK, quadrature PSK (QPSK), and 8-PSK. Note that for a given number of constellation points Q, the SNR penalty is always referenced to a single-user system with the same Q. As can be seen, increasing the constellation size results in a significant degradation in performance, especially for the four-user system. More importantly though, the degradation is quickly reduced as the number of antennas in increased. In fact, for $L = 3$, the performance degradation is less than 2 dB for all three modulation formats and for both the two- and four-user systems.

IV. Conclusions

In this letter, we have extended the results of [1] for the joint detection of several fading cochannel signals with use of a diversity antenna array. In particular, we derived a fully analytical expression for the pairwise-error probability and the union bound on symbol-error rate. Using the asymptotic forms of the analytical expressions, we confirmed several results obtained numerically in the previous paper. We also obtained several new results, such as the fact that a single user with poor channel estimation degrades even those users with perfect channel estimation. We have also quantified the performance loss as the number of users increases and as the constellation density increases, and we showed that the losses shrink dramatically as the number of antennas is increased.

References

[1] S. J. Grant and J. K. Cavers, "Performance enhancement through joint detection of cochannel signals using diversity arrays," *IEEE Trans. Commun.*, vol. 46, pp. 1038–1049, Aug. 1998.
[2] P. DeRusso, R. Roy, and C. Close, *State Variables for Engineers*. New York: Wiley, 1965.
[3] S. J. Grant and J. K. Cavers, "Multiuser channel estimation for detection of cochannel signals," in *Proc. IEEE ICC'99*, Vancouver, BC, Canada, June 6–10, 1999.