A Unified Approach to the Performance Analysis of Digital Communication over Generalized Fading Channels

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Presented here is a unified approach to evaluating the error-rate performance of digital communication systems operating over a generalized fading channel. What enables the unification is the recognition of the desirable form for alternate representations of the Gaussian and Marcum Q-functions that are characteristic of error-probability expressions for coherent, differentially coherent, and noncoherent forms of detection. It is shown that in the largest majority of cases, these error-rate expressions can be put in the form of a single integral with finite limits and an integrand composed of elementary functions, thus readily enabling numerical evaluation.

Keywords— *Communication channels, differential phase shift keying, digital communication, dispersive channels, diversity methods, fading channels, frequency shift keying, phase shift keying, signal detection.*

I. INTRODUCTION

Using alternate representations of classic functions arising in the error-probability analysis of digital communication systems (e.g., the Gaussian Q-function and the Marcum Q-function), more than four decades of contributions made by hundreds of authors dealing with error-probability performance over generalized fading channels are now able to be unified under a common framework.[1] The unified approach allows previously obtained results to be simplified both analytically and computationally and new results to be obtained for special cases that heretofore resisted solution in a simple form. The coverage is extremely broad in that coherent, differentially coherent, and noncoherent communication systems are all treated, as well as a large variety of fading channel models typical of communication links of practical interest. For each combination of communication (modulation/detection) type and channel fading model, the average bit error rate (BER) or symbol error rate (SER) of the system is described and represented by an expression that is in a form that can be readily evaluated. In many cases, the result is obtainable as a closed-form expression, while in other cases, it takes on the form of a single integral with finite limits and an integrand composed of elementary (exponential and trigonometric) functions.[2] All cases considered correspond to real practical channels, and the expressions obtained can be readily evaluated numerically. Due to space constraints and the wide variety of communication types and fading channels to which the unified approach applies, we have chosen to omit such numerical results from this paper. These will, however, be presented in a forthcoming textbook [37] and journal papers [21], [34], [35] by the authors. Applications of the generic results include satellite, terrestrial, and maritime communications, single and multicarrier code division multiple access (CDMA), two-dimensional (space–path) diversity, and error-correction coded communications.

II. TYPES OF COMMUNICATION

The unified approach to be described allows for the performance evaluation of systems characterized by a large variety of modulation/detection combinations. Letting $\tilde{s}(t) = A_m e^{j(2\pi f_m t + \phi_m)}$ denote the generic complex baseband transmitted signal in the mth transmission interval $(m-1)T \le t \le mT$, then a summary of these various digital communication types is given in Table 1.

III. TYPES OF FADING CHANNELS

Aside from applying to a wide variety of digital communication system types, the versatility of the unified approach will allow evaluation of average BER for a host of multipath fading channel types typical of practical communication

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[1] A small sample of these contributions, which in a broad sense are pertinent to what we present here, can be found in [1]–[54]. For a more detailed list of references that are specifically pertinent to each of the issues addressed in this paper, see [14], [21], [25], [34], and [35].

[2] In some instances, a second Gauss–Hermite quadrature integral [38, (25.4.46)] may be needed.

Table 1 Modulation/Detection Types

Detection Type	Modulation	Complex Baseband Signal Attributes
Coherent	M-ary amplitude-shift-keying (M-ASK) or M-ary amplitude modulation (M-AM)	$A_m = \pm(2k-1)A;\quad k = 1,2,\ldots,M/2$ $f_m = 0,\quad \phi_m = 0$
	Quadrature amplitude-shift-keying (QASK) or quadrature amplitude modulation (QAM)	$A_m = [\pm(2k-1) \pm j(2l-1)]A;\quad k,l = 1,2,\ldots,\sqrt{M}/2$ $f_m = 0,\quad \phi_m = 0$
	M-ary phase-shift-keying (M-PSK)	$A_m = A,\quad f_m = 0,$ $\phi_m = (2i-1)\frac{\pi}{M};\quad i = 1,2,\ldots,M$
	Binary frequency-shift-keying (BFSK) (orthogonal or nonorthogonal)	$A_m = A,\quad \phi_m = 0$ $f_m = \pm\Delta f;$ $\Delta f = \begin{cases} 1/2T \text{ (orthogonal)} \\ .715/T \text{ (minimum correlation)} \end{cases}$ $\phi_m = 0$
Differentially Coherent	M-ary phase-shift-keying (M-PSK) • Classical (two-symbol observation) • Multiple (more than two symbol observation)	$A_m = A,\quad f_m = 0,$ $\phi_m = (2i-1)\frac{\pi}{M};\quad i = 1,2,\ldots,M$
Noncoherent	Binary frequency-shift-keying (BFSK) (orthogonal or nonorthogonal)	$A_m = A,\quad \phi_m = 0$ $f_m = \pm\Delta f;\quad \Delta f = 1/T$ (orthogonal)

environments. A summary of these various fading channel models and the environments to which they apply is given in Table 2.

IV. Types of Reception

The most general model for the reception of digital signals transmitted through a slowly varying fading medium is a multilink channel in which the transmitted signal is received over L_p separate channels (Fig. 1). In this figure, $\{\tilde{r}_l(t)\}$ is the set of L_p received replicas of the complex transmitted signal, with l the channel index and $\{\alpha_l\}, \{\theta_l\}, \{\tau_l\}$ the corresponding sets of random path amplitudes, phases, and delays, respectively. Because of the slow fading assumption, we assume that the elements of the sets are all constant over the data symbol interval. We assume that these sets are mutually independent. The fading amplitude on each of these channels is assumed to be a time-invariant random variable (RV) with a known probability density function (pdf). While it is more typical than not to assume independent, identically distributed (i.i.d.) fading among the multichannels, the multichannel model that we shall consider is sufficiently general to include the case where the different channels are correlated as well as nonidentically distributed. We call this type of multilink channel a *generalized fading channel*. In the case of the latter, two situations are possible: either the channel fading probability distributions all come from the same family but have different average powers—i.e., the *power delay profile* (PDP) or alternately the *multipath intensity profile* (MIP) across the channels is nonuniform—or more generally, the channel fading probabilities come from different distribution families. Last, with regard to the delays, the first channel is assumed to be the reference channel whose delay $\tau_1 = 0$. Without loss of generality, we order the delays such that $\tau_1 < \tau_2 < \cdots < \tau_{L_p}$. With such a general model as the above, we are able to handle a large variety of individual channel descriptions and their associated diversity types such as a) space, b) angle, c) polarization, d) frequency, e) multipath, etc. A description of these and others is presented in [3, pp. 238–239].

One special case of the above generic fading channel model on which we shall primarily focus our attention corresponds to multipath radio propagation wherein the fading is classified according to its selectivity. In the case of *frequency nonselective* fading, wherein the symbol time of the digital modulation is large compared to the maximum delay spread of the channel, there exists only a single resolvable path resulting in single channel reception ($L_p = 1$). The receiver for such a communication system can perform coherent, differentially coherent, or noncoherent detection.

When the fading environment is such that the maximum delay spread of the channel is large compared to the symbol time, i.e., *frequency selective* fading, then there exist multiple ($L_p > 1$) resolvable paths (the maximum number of which is determined by the ratio of the maximum delay spread to the symbol time) resulting in multiple channel reception.

For the generic case of multichannel reception, diversity combining can be employed at the receiver to improve signal-to-noise ratio (SNR) and thus average BER performance. The particular types of diversity combining that are

Table 2 Fading Channel Types and Their Environment

Channel Type	Description and Environment	Probability Density Function (PDF) of Fading Amplitude, α and SNR/symbol, γ
Rayleigh [40]	Mobile systems with no LOS path between transmitter and receiver antennas, propagation of reflected and refracted paths through troposphere and ionosphere, ship-to-ship radio links [41].	$p_r(\alpha;\Omega)=\frac{2\alpha}{\Omega}\exp\left(-\frac{\alpha^2}{\Omega}\right);\quad \alpha\geq 0$ $p_r(\gamma;\bar{\gamma})=\frac{1}{\bar{\gamma}}\exp\left(-\frac{\gamma}{\bar{\gamma}}\right);\quad \gamma\geq 0$ $\Omega=\overline{\alpha^2}$ = average fading power $\bar{\gamma}$ = average SNR per symbol
Nakagami-q (Hoyt) (spans range from one-sided Gaussian $(q=0)$ to Rayleigh $(q=1)$) [42, Eq. (52)], [43]	Satellite links subject to strong ionospheric scintillation [44]	$p_q(\alpha;\Omega,q)=\frac{(1+q^2)\alpha}{q\Omega}\exp\left(-\frac{(1+q^2)^2\alpha^2}{4q^2\Omega}\right)I_0\left(\frac{(1-q^4)\alpha^2}{4q^2\Omega}\right);\quad \alpha\geq 0$ $p_q(\gamma;\bar{\gamma},q)=\frac{(1+q^2)}{2q\bar{\gamma}}\exp\left(-\frac{(1+q^2)^2\gamma}{4q^2\bar{\gamma}}\right)I_0\left(\frac{(1-q^4)\gamma}{4q^2\bar{\gamma}}\right);\quad \gamma\geq 0$
Nakagami-n (Rice) (spans range from Rayleigh $(n=0)$ to no fading $(n=\infty)$) [42, Eq. (50)], [45]	Related to Rician K factor $(n^2=K)$ – propagation paths consisting of one strong direct LOS component and many random weaker components – microcellular urban and suburban land mobile, picocellular indoor and factory environments [46].	$p_n(\alpha;\Omega,n)=\frac{2(1+n^2)e^{-n^2}\alpha}{\Omega}\exp\left(-\frac{(1+n^2)\alpha^2}{\Omega}\right)I_0\left(2n\alpha\sqrt{\frac{1+n^2}{\Omega}}\right);\quad \alpha\geq 0$ $p_n(\gamma;\bar{\gamma},n)=\frac{(1+n^2)e^{-n^2}}{\bar{\gamma}}\exp\left(-\frac{(1+n^2)\gamma}{\bar{\gamma}}\right)I_0\left(2n\sqrt{\frac{(1+n^2)\gamma}{\bar{\gamma}}}\right);\quad \gamma\geq 0$
Nakagami-m (spans range from one-sided Gaussian $(m=1/2)$, Rayleigh $(m=1)$ to no fading $(m=\infty)$) [42, Eq. (11)]	Often best fit to land mobile [47], indoor mobile multipath propagation as well as ionospheric radio links.	$p_m(\alpha;\Omega,m)=\frac{2m^m\alpha^{2m-1}}{\Omega^m\Gamma(m)}\exp\left(-\frac{m\alpha^2}{\Omega}\right);\quad \alpha\geq 0$ $p_m(\gamma;\bar{\gamma},m)=\frac{m^m\gamma^{m-1}}{\bar{\gamma}^m\Gamma(m)}\exp\left(-\frac{m\gamma}{\bar{\gamma}}\right);\quad \gamma\geq 0$
Log normal shadowing [3, Sect. 2.4]	Caused by terrain, buildings, trees – urban land mobile systems, land mobile satellite systems [48].	$p_\sigma(\gamma;\mu,\sigma)=\frac{10/\ln 10}{\sqrt{2\pi}\sigma\gamma}\exp\left[-\frac{(10\log_{10}\gamma-\mu)^2}{2\sigma^2}\right];\quad \gamma\geq 0$ μ (dB) and σ (dB) are mean and standard deviation of $10\log_{10}\gamma$
Composite gamma/log normal [3, Sect. 2.4.2]	Nakagami-m multipath fading superimposed on log normal shadowing. Congested down-	$p_{m\sigma}(\gamma;m,\mu,\sigma)=\int_0^\infty \frac{m^m\gamma^{m-1}}{\Omega^m\Gamma(m)}\exp\left(-\frac{m\gamma}{\Omega}\right)$ $\times\frac{10/\ln 10}{\sqrt{2\pi}\sigma\Omega}\exp\left[-\frac{(10\log_{10}\gamma-\mu)^2}{2\sigma^2}\right]d\Omega;\quad \gamma\geq 0$

practical depend on the characteristics of the modulation and their associated detection. For coherent detection, the optimum form of diversity combining is *maximal ratio combining* (MRC), which is implemented in the form of a RAKE receiver [1], [2] (see Fig. 2). Such an implementation requires knowledge of the channel fading parameters, which is typically obtained from measurements made on the channel. Aside from its superiority of performance, the RAKE receiver is well suited to equal as well as unequal energy signals such as M-AM, M-QAM, or, for that

Table 2 *(Continued.)* Fading Channel Types and Their Environment

	town areas with slow-moving pedestrians and vehicles. Also in land mobile systems subject to vegetative and/or urban shadowing [49].	
Combined (time-shared) shadowed/ unshadowed [50]	Convex combination of unshadowed multipath and a composite multipath/ shadowed fading. Land mobile satellite systems [50].	$p_{AK\sigma}(\gamma; A, \bar{\gamma}^u, K, \mu^s, \sigma^s) = (1-A)p_n(\gamma; \bar{\gamma}^u, \sqrt{K}) + A p_{m\sigma}(\gamma; 1, \mu^s, \sigma^s)$ A is time-share factor; μ^s (dB) and σ^s (dB) are mean and standard deviation of $10\log_{10}\gamma$ during shadowed fraction of time; $\bar{\gamma}^u$ is average SNR/symbol during unshadowed fraction of time.

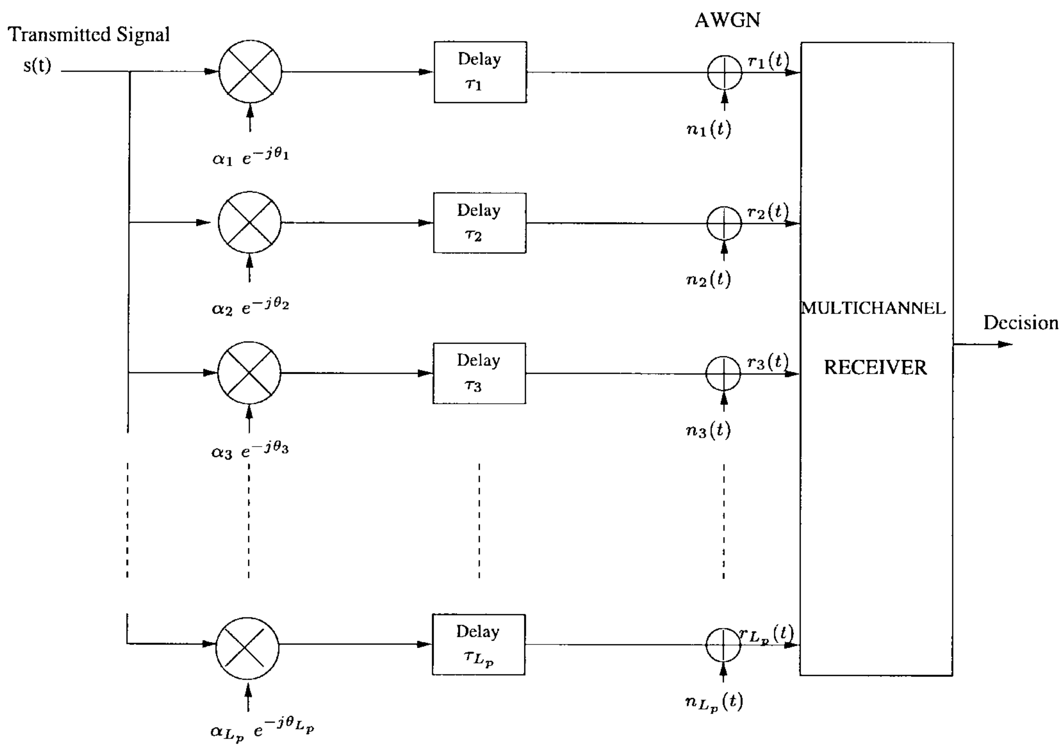

Fig. 1. Multilink channel model.

matter, any other amplitude/phase modulation. A simpler but suboptimum diversity combining technique is called *equal gain combining* (EGC) whose implementation has the advantage of not requiring knowledge of the channel fading amplitudes. Since unequal energy signals such as M-AM and M-QAM would require amplitude knowledge for automatic gain control (AGC) purposes, the EGC diversity technique should only be used with equal energy, i.e., constant envelope signals such as M-PSK [3, Sect. 5.5.4].

For differentially coherent and noncoherent detection, MRC is not practical since channel phase estimates are needed for its implementation. If in fact it were possible to estimate the channel phases on each path, then the reasons for employing differentially coherent and noncoherent detection would become mute, and instead one should resort to coherent detection since it results in superior performance. In view of this observation, the most appropriate form of diversity combining for these types of receivers is postdetection EGC [3, Sect. 5.5.6] (see Figs. 3 and 4).

With the foregoing material as background, we are now prepared to delve into the mechanisms that will allow the evaluation of the performance of such systems to be unified under a common framework.

V. Alternate Representations of the Gaussian and Marcum Q-Functions

A. The Gaussian Q-Function

The classical definition of the Gaussian Q-function (probability integral) is given by

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy. \tag{1}$$

In problems dealing with performance evaluation for coherent detection over fading channels, the conditional BER is expressed in terms of (1) where the argument x of the function is typically proportional to the square root of the instantaneous SNR, which itself depends on the random

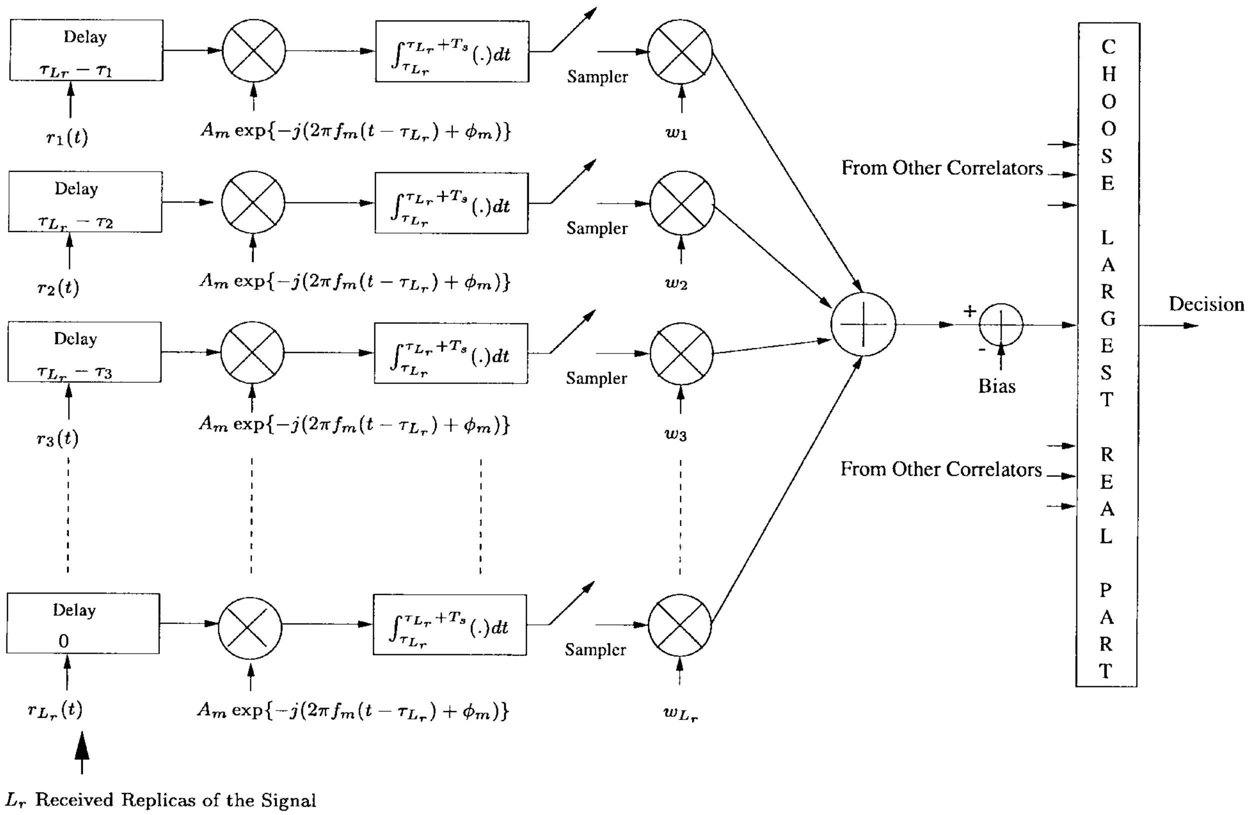

Fig. 2. Coherent multichannel receiver structure. The weights w_l are such that $w_l = (\alpha_l e^{j\theta_l})/N_l$ $(l = 1, 2, \cdots, L_r)$, and the bias is set equal to $\sum_{l=1}^{L_r} \alpha_l^2/N_l E_m$ for maximal-ratio combining; and $w_l = e^{j\theta_l}$ $(l = 1, 2, \cdots, L_r)$, and the bias is set equal to zero for equal-gain combining.

fading amplitudes of the various paths. To evaluate the average BER, one must then average over the statistics of the fading amplitude random variables. Since in the definition of (1) the argument x appears in the lower limit of the integral, it is analytically difficult to perform such averages. Rather, what would be desirable would be an integral form in which the limits were independent of the argument x (preferably finite from a computational standpoint) and an integrand that is exponential (preferably Gaussian) in the argument x.

A number of years ago, Craig [4] cleverly showed that the evaluation of average probability of error for the two-dimensional additive white Gaussian noise (AWGN) channel could be considerably simplified by choosing the origin of coordinates for each decision region as that defined by the *signal* vector as opposed to using a fixed coordinate system origin for all decision regions derived from the *received* vector. A by-product of this work was an alternate definite integral form for the Gaussian Q-function, which had the desirable properties mentioned above.[3] In particular

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta, \quad x \geq 0. \quad (2)$$

We herein refer to this form of the Gaussian Q-function as the *preferred* form since, as we shall see shortly, it simplifies the analysis and evaluation of average BER by allowing the averaging of the random parameters (fading amplitudes) to be performed inside the integral (in closed form for many cases) with a final integration on the variable θ performed at the end.

An interesting property of the form in (2) can be immediately obtained by inspection of the integrand. In particular, the maximum of the integrand occurs at the upper limit, i.e., for $\theta = \pi/2$. Thus, replacing the integrand by its maximum value, namely, $\exp(-x^2/2)$, immediately gives the upper bound

$$Q(x) \leq \frac{1}{2}\exp(-x^2/2), \quad x \geq 0 \quad (3)$$

which is the well-known Chernoff bound.

An interesting extension of the alternate representation in (2) can be obtained for the two-dimensional Gaussian Q-function, which has the classical form

$$Q(x, y; \rho) = \int_x^\infty \int_y^\infty \frac{1}{2\pi\sqrt{1-\rho^2}} \times \exp\left(-\frac{z_1^2 + z_2^2 - 2\rho z_1 z_2}{2(1-\rho^2)}\right) dz_1\, dz_2. \quad (4)$$

As was the case for (1), this form is undesirable in applications where additional statistical averaging must be performed over the arguments x, y of the function. In [6],

[3] This form of the Gaussian Q-function was earlier implied in the work of Pawula *et al.* [5] and Weinstein [54].

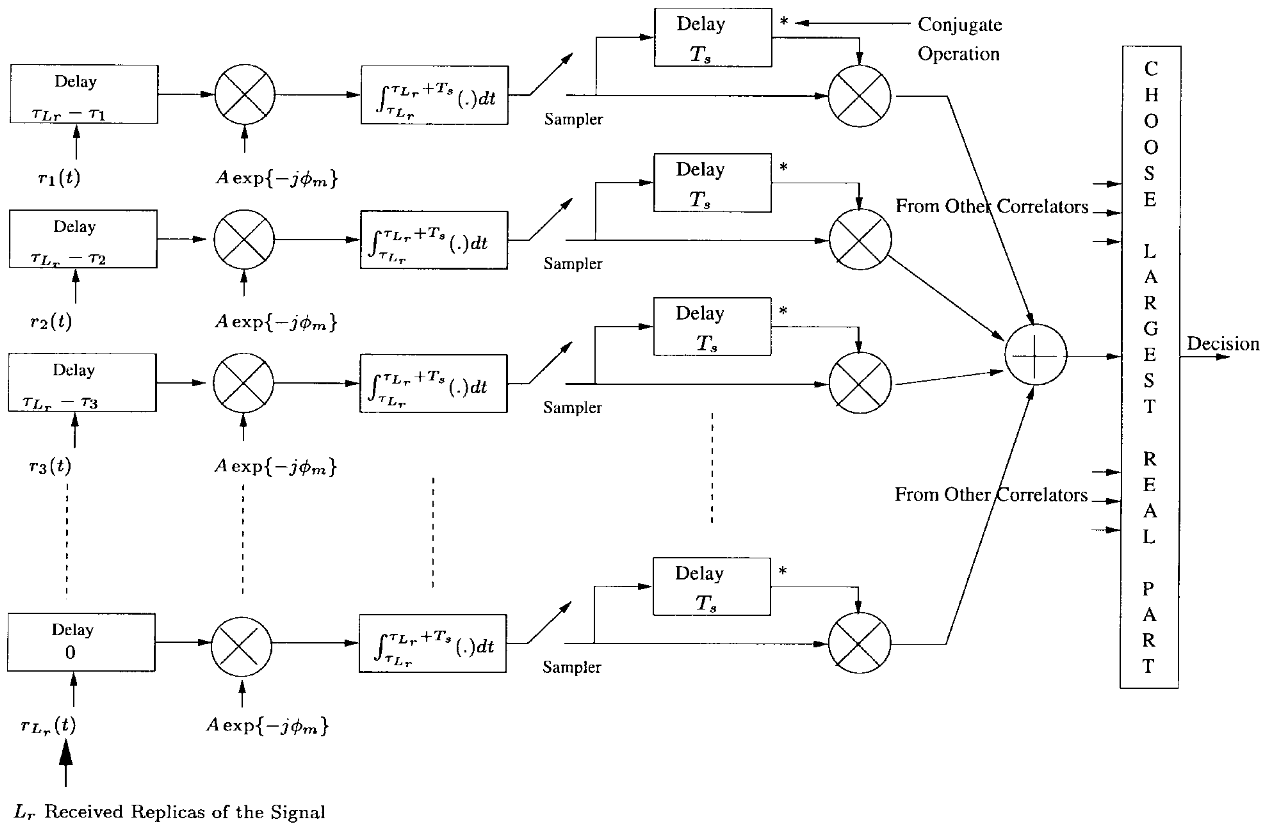

Fig. 3. Differentially coherent multichannel receiver structure.

Simon found a new representation for $Q(x,y;\rho)$ in the preferred form, namely

$$\begin{aligned}Q(x,y;\rho) &= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}-\tan^{-1}\frac{y}{x}} \frac{\sqrt{1-\rho^2}}{1-\rho\sin 2\theta} \\ &\quad\times\exp\left\{-\frac{x^2}{2}\frac{1-\rho\sin 2\theta}{(1-\rho^2)\sin^2\theta}\right\}d\theta \\ &\quad+\frac{1}{2\pi}\int_0^{\tan^{-1}\frac{y}{x}}\frac{\sqrt{1-\rho^2}}{1-\rho\sin 2\theta} \\ &\quad\times\exp\left\{-\frac{y^2}{2}\frac{1-\rho\sin 2\theta}{(1-\rho^2)\sin^2\theta}\right\}d\theta, \\ &\qquad x,y\geq 0. \qquad (5)\end{aligned}$$

A special case of (5) is of particular interest, namely, when $x = y$ and $\rho = 0$. For this case, (5) simplifies to

$$Q^2(x) = \frac{1}{\pi}\int_0^{\pi/4}\exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta, \quad x \geq 0. \qquad (6)$$

Comparing (6) with (2), we see that the square of the Gaussian Q-function has the same integrand as the Gaussian Q-function itself but integrated only over *half* the interval. As we shall see, the result in (6) is particularly useful in evaluation of average SER for M-QAM transmitted over fading channels.

B. The Marcum Q-Function

The first-order Marcum Q-function [7] is classically defined as

$$Q_1(\alpha,\beta) = \int_\beta^\infty x\exp\left[-\left(\frac{x^2+\alpha^2}{2}\right)\right]I_0(\alpha x)\,dx. \qquad (7)$$

In problems dealing with performance evaluation for differentially coherent and noncoherent detection over fading channels, the conditional BER is expressed in terms of (7), where, as in the previous discussion, the arguments α, β of the function are typically both proportional to the square root of the instantaneous SNR, which itself depends on the random fading amplitudes of the various paths. To evaluate the average BER, one must again average over the statistics of the fading amplitude random variables, and thus (7) has the same undesirability as (1). The natural question to ask is: Is it possible to arrive at a representation of the Marcum Q-function in the so-called preferred form, i.e., one where the limits of the integral are independent of the arguments of the function (and hopefully also finite) and the integrand is a Gaussian function of these arguments? Before answering this question, we make one more important observation. While it is true, as mentioned above, that the arguments α, β of the Marcum Q-function typically both depend on the random fading amplitudes of the various paths, their ratio is independent of the instantaneous SNR and depends only on the modulation/detection type. With this in mind, we define $\zeta = \alpha/\beta$ which is a nonrandom parameter that requires no statistical averaging and is in many cases simply

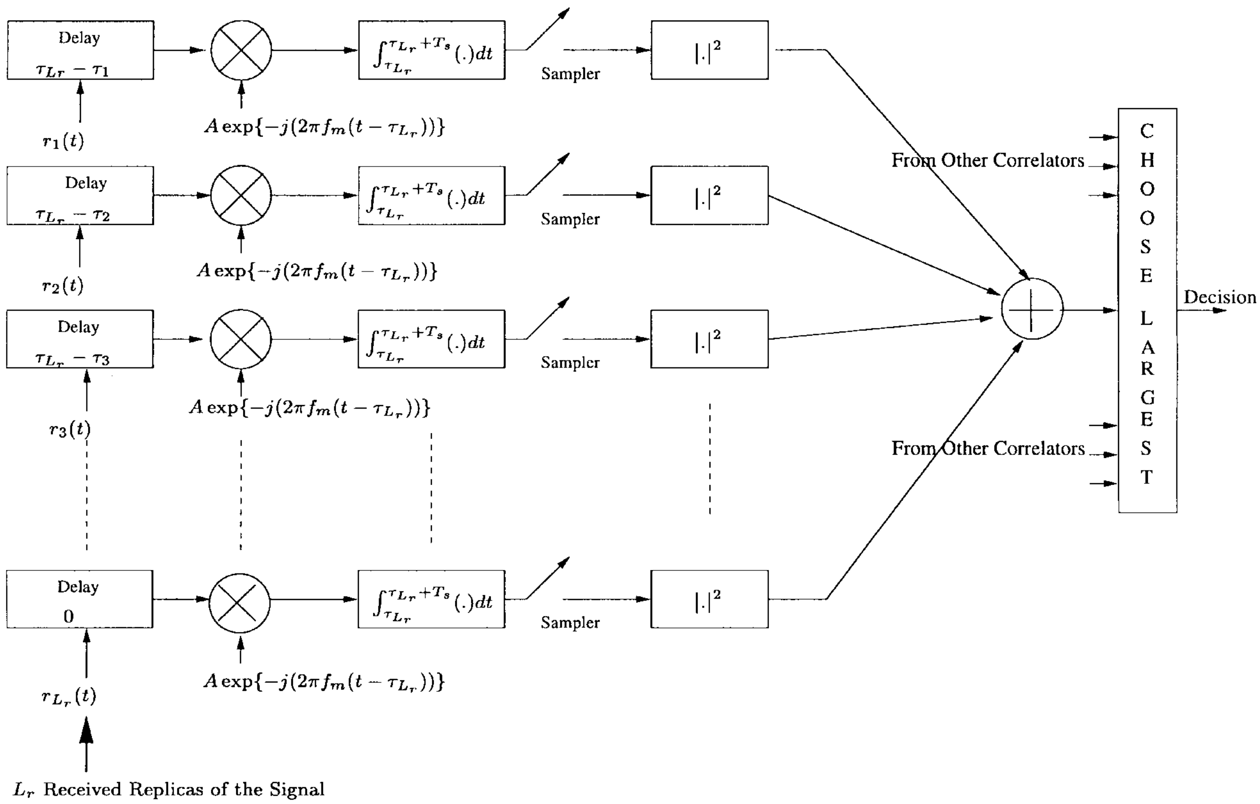

Fig. 4. Noncoherent multichannel receiver structure.

a number (more about this later, when we consider specific modulation/detection examples). Thus, substituting $\beta\zeta$ for α in (7), we reduce the definition to a single statistical argument β, i.e.,

$$Q_1(\beta\zeta,\beta) = \int_\beta^\infty x\exp\left[-\left(\frac{x^2+(\beta\zeta)^2}{2}\right)\right] I_0(\beta\zeta x)\,dx. \tag{8}$$

Having done this, we are now in a position to offer a positive answer to the above question. Using the infinite series representation [8, p. 153] of the Marcum Q-function and the integral representation of the kth order modified Bessel function of the first kind, namely, $I_x(z) = (1/2\pi)\int_{-\pi}^{\pi}(-je^{-j\theta})^k e^{-z\sin\theta}\,d\theta$, it was shown in [52] that for $\alpha < \beta$, or equivalently in [9] for $0 \le \zeta < 1$

$$\begin{aligned} Q_1(\beta\zeta,\beta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1+\zeta\sin\theta}{1+2\zeta\sin\theta+\zeta^2}\right] \\ &\quad\times\exp\left(-\frac{\beta^2}{2}[1+2\zeta\sin\theta+\zeta^2]\right)d\theta, \\ &\qquad 0\le\zeta<1. \end{aligned} \tag{9}$$

Observe that the integration limits in (9) are finite and independent of the random argument β, and the integrand is Gaussian in this same argument. Similarly, for $\beta < \alpha$, defining now the parameter $\zeta = \beta/\alpha$, then substituting $\alpha\zeta$ for β in (7) to reduce the classical definition to a single statistical argument (now α), it was shown in [9] and [52] that[4]

$$\begin{aligned} Q_1(\alpha,\alpha\zeta) &= 1+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{\zeta^2+\zeta\sin\theta}{1+2\zeta\sin\theta+\gamma^2}\right] \\ &\quad\times\exp\left(-\frac{\alpha^2}{2}[1+2\zeta\sin\theta+\zeta^2]\right)d\theta, \\ &\qquad 0\le\zeta<1. \end{aligned} \tag{10}$$

Simple checks on the validity of the results in (9) and (10) immediately produce

$$Q_1(0,\beta)=\exp(-\beta^2/2), \quad Q_1(\alpha,0)=1. \tag{11}$$

Also, in the same manner as was done for the Gaussian Q-function, one can immediately obtain upper and lower "Chernoff-type" bounds on the Marcum Q-function. In particular, observing that the maximum and minimum of the integrand in (10) occurs for $\theta = -\pi/2$ and $\theta = \pi/2$, respectively, then replacing the integrand by its maximum

[4] Although it appears from (10) that the Marcum Q-function can exceed unity, we note that the integral portion of this equation is always less than or equal to zero. Furthermore, the special case of $\alpha = \beta (\zeta = 1)$, which has limited interest in communication performance applications, has the closed-form result $Q_1(\alpha,\alpha) = [1+\exp(-\alpha^2)I_0(\alpha^2)]/2$ [39, (A-3-2)]. It should also be noted that the results in (9) and (10) can also be obtained from the work of Pawula dealing with the relation between the Rice Ie-function and the Marcum Q-function [53]. In particular, equating [53, (2a) and (2c)] and using the integral representation of the zero-order Bessel function as above in the latter of the two equations, one can, with an appropriate change of variables, arrive at (9) and (10) of this paper.

and minimum values gives

$$\frac{1}{1+\zeta}\exp\left(-\frac{\beta^2(1+\zeta)^2}{2}\right) \le Q_1(\beta\zeta,\beta) \le \frac{1}{1-\zeta}\exp\left(-\frac{\beta^2(1-\zeta)^2}{2}\right), \quad 0 \le \zeta \triangleq \alpha/\beta < 1 \quad \text{(12a)}$$

which in view of (11) are asymptotically tight as $\zeta \to 0$. Similarly, for $\beta < \alpha$, the lower bound becomes[5]

$$1 - \frac{\zeta}{1-\zeta}\exp\left(-\frac{\alpha^2(1-\zeta)^2}{2}\right) \le Q_1(\alpha,\alpha\zeta), \quad 0 \le \zeta \triangleq \beta/\alpha < 1. \quad \text{(12b)}$$

Last, we point out that the integrals in (9) and (10) can be put in a more reduced form, wherein the limits of integration are $0, \pi$ rather than $-\pi, \pi$. The necessary changes to the integrand are: replace $\sin\theta$ by $\cos\theta$, replace ζ by $-\zeta$, and multiply the entire integrand by two. From the standpoint of performance evaluation, there is no particular advantage gained by reducing the range of integration, and hence we continue to use the forms already presented in all that follows.

The desirable form of the representation for the first-order Marcum Q-function given in (9) and (10) can also be obtained for the generalized (lth-order) Marcum Q-function defined by

$$Q_l(\alpha,\beta) = \frac{1}{\alpha^{l-1}}\int_\beta^\infty x^l \exp\left[-\left(\frac{x^2+\alpha^2}{2}\right)\right] I_{l-1}(\alpha x)\,dx. \quad \text{(13)}$$

In particular, starting with the series representations for the generalized Marcum Q-function and again making use of the integral representation of the $l-1$st order modified Bessel function of the first kind, the following pair of relations was derived in [9] and [52]:

$$Q_l(\beta\zeta,\beta) = \frac{1}{2\pi}\times\int_{-\pi}^{\pi}\left[\frac{\zeta^{-(l-1)}\left(\cos\left[(l-1)\left(\theta+\frac{\pi}{2}\right)\right] - \zeta\cos\left[l\left(\theta+\frac{\pi}{2}\right)\right]\right)}{1+2\zeta\sin\theta+\zeta^2}\right] \times \exp\left(-\frac{\beta^2}{2}\left[1+2\zeta\sin\theta+\zeta^2\right]\right)d\theta, \quad 0^+ \le \zeta = \alpha/\beta < 1 \quad \text{(14a)}$$

$$Q_l(\alpha,\alpha\zeta) = 1 - \frac{1}{2\pi}\times\int_{\pi}^{\pi}\left[\frac{\zeta^{l}\left(\cos\left[l\left(\theta+\frac{\pi}{2}\right)\right] - \zeta\cos\left[(l-1)\left(\theta+\frac{\pi}{2}\right)\right]\right)}{1+2\zeta\sin\theta+\zeta^2}\right] \times \exp\left(-\frac{\alpha^2}{2}\left[1+2\zeta\sin\theta+\zeta^2\right]\right)d\theta, \quad 0 \le \zeta = \beta/\alpha < 1. \quad \text{(14b)}$$

With the above mathematical tools in hand, we are now in a position to demonstrate how the performance of coherent, differentially coherent, and noncoherent communication systems operating over generalized fading channels can be evaluated both analytically and numerically in terms of finite integrals with simple integrands, which in some cases can be reduced to closed-form solutions.

[5] Since the maximum of the integrand in (10), which occurs at $\theta = \pi/2$, would exceed unity, then replacing the integrand by this maximum value gives a useless upper bound.

VI. Coherent Multichannel Detection of Digital Signals

A. Multichannel Mathematical Model

In keeping with the multichannel representation of Fig. 1, after passing through the fading channel, each replica of the signal is perturbed by AWGN with a single-sided power spectral density $N_l; l = 1, 2, \cdots, L_p$ (W/Hz). The AWGN is assumed to be statistically independent from channel to channel and independent of the fading amplitudes $\{\alpha_l\}$. Relating Fig. 1 to the channels described by Table 2, the fading amplitude of the lth channel is an RV with mean square value $\bar{\alpha}_l^2 = \Omega_l$ and whose pdf is any of those described in the table. Mathematically speaking, for the generic communication signal described in Section II, the receiver is provided the set of complex baseband received signals

$$\tilde{r}_l(t) = \alpha_l e^{-j\theta_l}\tilde{s}(t-\tau_l) + \tilde{n}_l(t) = A_m\alpha_l e^{j(2\pi f_m(t-\tau_l)+\phi_m-\theta_l)} + \tilde{n}_l(t), \quad l = 1, 2, \cdots, L_p \quad \text{(15)}$$

where $\tilde{n}_l(t)$ denotes the equivalent complex baseband AWGN for the lth channel with single-sided power spectral density $2N_l$. The instantaneous SNR per symbol of the lth channel is defined as $\gamma_l \triangleq \alpha_l^2 E_s/N_l$ where E_s denotes the average symbol energy and for a given type of signaling scheme can be related to the amplitude A_m introduced in Section II.

One common example of a multichannel that is typical of a wide class of radio propagation environments is the multipath channel, which can be modeled as a linear filter characterized by the complex-valued low-pass equivalent impulse response [10]–[12]

$$h(t) = \sum_{l=1}^{L_p}\alpha_l e^{-j\theta_l}\delta(t-\tau_l) \quad \text{(16a)}$$

where $\delta(\cdot)$ is the Dirac delta function. The difference between adjacent delays, i.e., $\tau_l - \tau_{l-1}$, is most often modeled as being constant and equal to the symbol time, in which case the linear filter takes on the form of a uniformly spaced tapped delay line with $L_p - 1$ taps. For the special case of the multipath channel defined by (16a), the single received signal would take the form

$$\tilde{r}(t) = \sum_{l=1}^{L_p}\alpha_l e^{-j\theta_l}\tilde{s}(t-\tau_l) + \tilde{n}(t) = A_m\sum_{l=1}^{L_p}\alpha_l e^{j(2\pi f_m(t-\tau_l)+\phi_m-\theta_l)} + \tilde{n}(t) \quad \text{(16b)}$$

where now $\tilde{n}(t)$ represents the equivalent baseband complex noise associated with the single receiver and has power spectral density $2N_0$ W/Hz. As previously mentioned, in what follows we shall primarily focus on the multipath channel model of (16a) and associated received signal form of (16b), although the approach applies equally well to the other forms of the generic multichannel model and their associated diversity types.[6] This is tantamount to assuming a generic multichannel model with $N_l = N_0, l = 1, 2, \cdots, L_p$.

B. Average BER for Binary Signals

For binary signals and a receiver that implements diversity combining with ideal time and phase synchronization on each branch (i.e., perfect time delay $\{\tau_l\}$ and phase $\{\theta_l\}$ estimates), the conditional (on the fading amplitudes) BER is given by [13, p. 188]

$$P_b(E; \{\alpha_l\}) = Q\left(\sqrt{g\frac{2E_b}{N_0}\eta}\right) \tag{17}$$

where $g = 1$ for coherent binary phase shift keying (BPSK), $g = 1/2$ for coherent orthogonal binary frequency shift keying (BFSK), and $g = \frac{1}{2} + \frac{1}{3\pi}$ for coherent BFSK with minimum correlation. The parameter η is a function of the set of fading amplitudes $\{\alpha_l\}$ and has a form that depends on the type of diversity combining employed. That is, for MRC and perfect estimation of the fading amplitudes $\{\alpha_l\}$, we would have [3]

$$\eta = \sum_{l=1}^{L_r} \alpha_l^2 \triangleq \alpha_{\mathrm{MRC}} \tag{18}$$

whereas for EGC with no estimation of the fading amplitudes, we would have [3]

$$\eta = \frac{1}{L_r}\left(\sum_{l=1}^{L_r} \alpha_l\right)^2 \triangleq \frac{1}{L_r}\alpha_{\mathrm{EGC}}^2, \quad \alpha_{\mathrm{EGC}} = \sum_{l=1}^{L_r} \alpha_l. \tag{19}$$

In (18) and (19), $L_r \leq L_p$ is the *actual* number of combined paths in the RAKE receiver.[7] We also note that the results in (17) together with (18) or (19) also apply to diversity combining of L_r receivers of the same information-bearing signal transmitted over L_r frequency nonselective, slow fading channels.

To compute the average BER, we must statistically average (17) over the joint PDF of the fading amplitudes, i.e.,

$$P_b(E) = \int_0^\infty \cdots \int_0^\infty P_b(E; \{\alpha_l\}) p_\alpha (\alpha_1, \cdots, \alpha_{L_r})\, d\alpha_1 \cdots d\alpha_{L_r}. \tag{20}$$

If the fading amplitudes $\{\alpha_l\}$ are statistically independent (but not necessarily identically distributed), then (20) reduces to

$$P_b(E) = \int_0^\infty \cdots \int_0^\infty P_b(E; \{\alpha_l\}) p_{\alpha_l}(\alpha_1) \cdots p_{\alpha_{L_r}}(\alpha_{L_r})\, d\alpha_1 \cdots d\alpha_{L_r}. \tag{21}$$

1) Classical Solution: The classical solution to (21) is first to replace the L_r-fold average by a single average over η, i.e.,

$$P_b(E) = \int_0^\infty Q\left(\sqrt{g\frac{2E_b}{N_0}\eta}\right) p_\eta(\eta)\, d\eta. \tag{22}$$

Note that (22) does not require the independence assumption on the fading amplitudes and thus also applies to (20). Evaluation of (22) requires obtaining the pdf of the combined fading RV η. For the case where the fading amplitudes can be assumed independent, finding this pdf requires a convolution of the pdf's of the α_l and can often be quite difficult to evaluate, particularly if the pdf's of the α_l come from different distribution families. Even in the case where the pdf's of the α_l come from the same distribution family but have different average powers, i.e., other than a uniform power delay profile, evaluation of the pdf of η can still be quite difficult. To circumvent this difficulty, we now propose an alternate method of solution based on using the alternate representation of the Gaussian Q-function in (2).

2) Solution Based on Alternate Representation of the Gaussian Q-Function:

a) MRC with independent (but not necessarily identical) fading amplitudes: Combining (17) and (18) and using the alternate representation of the Gaussian Q-function of (2), the average BER of (21) can be expressed as

$$\begin{aligned} P_b(E) &= \int_0^\infty \cdots \int_0^\infty Q\left(\sqrt{g\frac{2E_b}{N_0}\sum_{l=1}^{L_r}\alpha_l^2}\right) \\ &\qquad \times p_{\alpha_1}(\alpha_1)\cdots p_{\alpha_{L_r}}(\alpha_{L_r})\, d\alpha_1 \cdots d\alpha_{L_r} \\ &= \int_0^\infty \cdots \int_0^\infty \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{g\frac{E_b}{N_0}\sum_{l=1}^{L_r}\alpha_l^2}{\sin^2\theta}\right) \\ &\qquad \times p_{\alpha_1}(\alpha_1)\cdots p_{\alpha_{L_r}}(\alpha_{L_r}) \\ &\qquad \times d\theta\, d\alpha_1 \cdots d\alpha_{L_r} \\ &= \frac{1}{\pi}\int_0^{\pi/2} \prod_{l=1}^{L_r} I_l(\bar{\gamma}_l, g, \theta)\, d\theta \end{aligned} \tag{23}$$

[6] We note that for the frequency-selective multipath channel case, the proper operation of the RAKE receiver requires that the transmitted signals be given an orthogonal basis.

[7] For MRC, the higher L_r is, the better the performance, and hence from this standpoint alone, the optimal value for L_r is L_p. However, L_r is typically chosen strictly less than L_p due to receiver complexity constraints. For EGC, L_r is also typically chosen strictly less than L_p; however, the reason for this choice here is not due solely to complexity constraints. In addition, under certain circumstances, increasing L_r may induce a "combining loss," and thus from a performance standpoint, the optimum value of L_r may not be L_p. Indeed, equal-weight combining of paths with very low average SNR degrades performance, since these paths will contribute mostly to noise [24]. Thus, it is better not to include these paths in the combining process.

Table 3 Evaluation of the Integrals $I_l(\bar{\gamma}_l, g, \theta)$

Channel Type	$I_l(\overline{\gamma}_l, g, \theta)$
Rayleigh	$\left(1+\frac{g\overline{\gamma}_l}{\sin^2\theta}\right)^{-1}$
Nakagami-*q* (Hoyt)	$\left(1+\frac{2g\overline{\gamma}_l}{\sin^2\theta}+\frac{4q_l^2 g^2(\overline{\gamma}_l)^2}{(1+q_l^2)^2\sin^4\theta}\right)^{-1/2}$
Nakagami-*n* (Rice)	$\frac{(1+n_l^2)\sin^2\theta}{(1+n_l^2)\sin^2\theta+g\overline{\gamma}_l}\exp\left(-\frac{n_l^2 g\overline{\gamma}_l}{(1+n_l^2)\sin^2\theta+g\overline{\gamma}_l}\right)$; $n_l^2 = K_l$ (K_l is Rician factor)
Nakagami-*m*	$\left(1+\frac{g\overline{\gamma}_l}{m_l\sin^2\theta}\right)^{-m_l}$
Log normal shadowing*	$\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N_p} H_{x_n}\exp\left(-\frac{10^{(\sqrt{2}\sigma_l x_n+\mu_l)/10}g}{\sin^2\theta}\right)$; x_n and H_{x_n} are zeros and weight factors of the N_p-order Hermite polynomial $H_{N_p}(\cdot)$ which are tabulated in [38, p. 924, Table (25.10)]; μ_l (dB) and σ_l (dB) are mean and stanadard deviation of $10\log_{10}\gamma_l$.
Composite gamma/ log normal*	$\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N_p} H_{x_n}\left(1+\frac{10^{(\sqrt{2}\sigma_l x_n+\mu_l)/10}g}{m_l\sin^2\theta}\right)^{-m_l}$
Combined (time-shared) shadowed/unshadowed**	$(1-A_l)\frac{(1+K_l)\sin^2\theta}{(1+K_l)\sin^2\theta+g\overline{\gamma}_l^{\,u}}\exp\left(-\frac{K_l g\overline{\gamma}_l}{(1+K_l)\sin^2\theta+g\overline{\gamma}_l^{\,u}}\right)+A_l\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N_p} H_{x_n}\left(1+\frac{10^{(\sqrt{2}\sigma_l^s x_n+\mu_l^s)/10}g}{\sin^2\theta}\right)^{-1}$; A_l is time-share factor; μ_l^s (dB) and σ_l^s (dB) are mean and standard deviation of $10\log_{10}\gamma_l$ during shadowed fraction of time; $\overline{\gamma}_l^{\,u}$ is average SNR/ symbol during unshadowed fraction of time.

*The expressions given for $I_l(\overline{\gamma}_l, g, \theta)$ are derived from a Gauss-Hermite quadrature integration rule and as such are approximate.

** The first part of the expression for $I_l(\overline{\gamma}_l, g, \theta)$ is exact and the second part having been derived from a Gauss-Hermite quadrature integration rule is approximate.

where

$$I_l(\bar{\gamma}_l, g, \theta) \triangleq \int_0^\infty \exp\left(-\frac{gE_b}{N_0\sin^2\theta}\alpha_l^2\right)p_{\alpha_l}(\alpha_l)\,d\alpha_l = \int_0^\infty \exp\left(-\frac{g\gamma_l}{\sin^2\theta}\right)p_{\gamma_l}(\gamma_l)\,d\gamma_l \tag{24}$$

and

$$\gamma_l \triangleq \frac{\alpha_l^2 E_b}{N_0}, \quad \bar{\gamma}_l \triangleq \frac{\Omega_l E_b}{N_0} \tag{25}$$

are, respectively, the instantaneous and average SNR per bit corresponding to the lth channel (or resolvable path). The form of the average BER in (23) is quite desirable in that the integrals $I_l(\bar{\gamma}_l, g, \theta)$ can be obtained in closed form with the help of known Laplace transforms or can alternately be efficiently computed using Gauss–Hermite quadrature integration. Thus, all that remains to compute is a *single integral (on θ) over finite limits*. The results of these evaluations for the considered fading channel distributions in Table 2 are obtained [14] with the aid of a number of definite integrals in [15] and are tabulated in Table 3. Last, for the special case where all L_r channels are identically distributed with the same average SNR per bit $\bar{\gamma}_1$, for all channels, then (22) simplifies further to

$$P_b(E) = \frac{1}{\pi}\int_0^{\pi/2}[I_1(\bar{\gamma}_1, g, \theta)]^{L_r}\,d\theta. \tag{26}$$

b) EGC with independent (but not necessarily identical) fading amplitudes: Combining (17) and (19) and using the alternate representation of the Gaussian Q-function of (2), the average BER of (21) can be expressed as

$$\begin{aligned}P_b(E) &= \int_0^{\infty}\cdots\int_0^{\infty} Q\left(\sqrt{g\frac{2E_b}{N_0 L_r}\left(\sum_{l=1}^{L_r}\alpha_l\right)^2}\right)\\ &\quad\times p_{\alpha_1}(\alpha_1)\cdots p_{\alpha_{L_r}}(\alpha_{L_r})\,d\alpha_1\cdots d\alpha_{L_r}\\ &= \int_0^{\infty}\cdots\int_0^{\infty}\frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{\frac{gE_b}{N_0L_r}\left(\sum_{l=1}^{L_r}\alpha_l\right)^2}{\sin^2\theta}\right)\\ &\quad\times p_{\alpha_1}(\alpha_1)\cdots p_{\alpha_{L_r}}(\alpha_{L_r})\\ &\quad\times\, d\theta\, d\alpha_1\cdots d\alpha_{L_r}. \end{aligned} \tag{27}$$

Unfortunately, for this type of diversity combining, we cannot represent the exponential in (27) as a product of exponentials each involving only a single α_l because of the presence of the $\alpha_k\alpha_l$ cross-product terms. Hence, we cannot partition the L_r-fold integral. Instead, we must return to the classical solution of Section VI-B1 but now use the alternate representation of the Gaussian Q-function. Letting $\alpha \triangleq \alpha_{\mathrm{EGC}}$ for simplicity of notation, we have from (2), (19), and (22)

$$\begin{aligned}P_b(E) &= \int_0^{\infty}\frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{gE_b}{N_0L_p\sin^2\theta}\alpha^2\right)d\theta\, p_\alpha(\alpha)\,d\alpha\\ &= \frac{1}{\pi}\int_0^{\pi/2}\int_0^{\infty}\exp\left(-\frac{A^2}{2\sin^2\theta}\alpha^2\right)p_\alpha(\alpha)\,d\alpha\,d\theta,\\ &\qquad A=\sqrt{\tfrac{2gE_b}{N_0L_r}}. \end{aligned}\tag{28}$$

Next, we represent $p_\alpha(\alpha)$ in terms of its inverse Fourier transform, i.e., the characteristic function, which, because of the independence assumption on the fading channel amplitudes, becomes

$$\begin{aligned}p_\alpha(\alpha) &\triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_\alpha(jv)e^{-jv\alpha}\,dv\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\prod_{l=1}^{L_r}\psi_{\alpha_l}(jv)\right]e^{-jv\alpha}\,dv.\end{aligned}\tag{29}$$

Substituting (29) into (28) gives

$$\begin{aligned}P_b(E) &= \frac{1}{2\pi^2}\int_0^{\pi/2}\int_{-\infty}^{\infty}\overbrace{\left[\prod_{l=1}^{L_r}\psi_{\alpha_l}(jv)\right]}^{\Psi(jv)}\\ &\quad\times\overbrace{\left[\int_0^{\infty}\exp\left(-\frac{A^2}{2\sin^2\theta}\alpha^2 - jv\alpha\right)d\alpha\right]}^{J(v,\theta)}\,dv\,d\theta.\end{aligned}\tag{30}$$

The integral $J(v,\theta)$ can be obtained in closed form by separately evaluating its real and imaginary parts, namely, [15, (3.896.4) and (3.896.3)]

$$\begin{aligned}&\int_0^{\infty}\exp\left(-\frac{A^2}{2\sin^2\theta}\alpha^2\right)\cos v\alpha\,d\alpha\\ &\quad=\sqrt{\frac{\pi\sin^2\theta}{2A^2}}\exp\left(-v^2\frac{\sin^2\theta}{2A^2}\right)\\ &\quad\triangleq X(\theta)\exp\left(-v^2\frac{\sin^2\theta}{2A^2}\right)\\ &\int_0^{\infty}\exp\left(-\frac{A^2}{2\sin^2\theta}\alpha^2\right)\sin v\alpha\,d\alpha\\ &\quad=\frac{\sin^2\theta}{A^2}v\exp\left(-v^2\frac{\sin^2\theta}{2A^2}\right){}_1F_1\left(\frac{1}{2};\frac{3}{2};v^2\frac{\sin^2\theta}{2A^2}\right)\\ &\quad\triangleq Y(v,\theta)\exp\left(-v^2\frac{\sin^2\theta}{2A^2}\right)\end{aligned}\tag{31}$$

where ${}_1F_1(\cdot;\cdot;\cdot)$ is the confluent hypergeometric function. Despite the fact that the product of characteristic functions $\Psi(jv)\triangleq W(v)\exp(j\Theta(v))$ in (30) is, in general, complex, the average BER is real; thus, it is sufficient to consider only the real part of the integrand in this equation. Last, using (31) in (30) and making the change of variables $x = v\sin\theta/\sqrt{2}A$, we obtain

$$P_b(E) = \frac{1}{2\pi^2}\int_0^{\pi/2}\int_{-\infty}^{\infty}\exp(-x^2)f\left(\frac{\sqrt{2}A}{\sin\theta}x,\theta\right)dx\,d\theta \tag{32}$$

where

$$\begin{aligned}f(v,\theta) &\triangleq \sqrt{X^2(\theta)+Y^2(v,\theta)}W(v)\\ &\quad\times\cos\left(\tan^{-1}\frac{Y(v,\theta)}{X(v,\theta)}+\Theta(v)\right)\end{aligned}\tag{33}$$

and the doubly infinite integral on x can be readily evaluated by the Gauss–Hermite quadrature formula

$$\int_{-\infty}^{\infty}\exp(-x^2)f\left(\frac{\sqrt{2}A}{\sin\theta}x,\theta\right)dx = \sum_{n=1}^{N_p}H_{x_n}f\left(\frac{\sqrt{2}A}{\sin\theta}x_n,\theta\right) \tag{34}$$

where $\{x_n\},\{H_{x_n}\}$ are zeros and weight factors of the N_p-order Hermite polynomial $H_{N_p}(x)$. These coefficients are tabulated in [38, p. 924, Table 25.10] for various polynomial orders. Typically, $N_p = 20$ is sufficient for excellent accuracy. Last, after substituting (32) into (30), what remains is a single integral on θ over finite limits.

While the solution for the average BER of the EGC receiver is indeed one step more complicated than that for the MRC receiver, i.e., one must evaluate a Gauss quadrature integral in addition to the finite limit integral on θ, we wish to remind the reader of the generality of our model, namely, each fading channel carries its own individual fading amplitude statistic. When contrasted with the true classical solution in the form of (22), which would require an L_r-fold convolution (itself an L_r-fold integral) or other means to obtain the pdf of the combined fading RV η

[17], the form of the solution as given by (30) together with (33) and (34) is considerably simpler. The full details of this approach for Nakagami-m distributed channels (paths) are given in [16]. Another approach, specifically for Rayleigh fading, which sometimes leads to closed-form solutions is discussed in [18].

c) MRC with correlated fading amplitudes: As discussed in [19] and [20], there are a number of real-life scenarios in which the assumption of independent paths is not valid. Along these lines, two correlation models have been proposed, namely, equal (constant) correlation and exponential correlation, each with its own advantages and disadvantages depending on the physics of the channel. Using these models along with a Nakagami-m distribution for the fading, several authors have analyzed special cases of the performance of such systems corresponding to specific modulation, detection, and diversity combining schemes. For example, Aalo [19] obtains the average BER for multichannel reception of coherent and noncoherent BFSK and coherent and differentially coherent BPSK using an MRC. Patenaude *et al.* [20] consider this same performance for postdetection EGC of the multichannel reception of orthogonal BFSK and differentially coherent BPSK (DPSK).

In this section, we obtain general results for the average BER of binary coherent modulations over equicorrelated and exponentially correlated Nakagami-m channels. Aside from allowing for many modulation/detection/diversity combining cases not previously treated, these general results as before provide in many cases much simpler forms for average BER expressions corresponding to the special cases treated in [19].

From (18) and (25), the total SNR per bit γ at the output of the MRC is given by

$$\gamma \triangleq \eta \frac{E_b}{N_0} = \sum_{l=1}^{L_r} \frac{\alpha_l^2 E_b}{N_0} = \sum_{l=1}^{L_r} \gamma_l. \tag{35}$$

It is shown in [19, (18)] that for the equicorrelated fading model, the pdf of γ is given by (36), shown at the bottom of the page, where $0 \leq \rho < 1$ is the envelope correlation coefficient assumed to be the same between all channel pairs.[8] As mentioned in [19, Sect. II-A], such a correlation model may approximate closely placed diversity antennas. Similarly for the exponential correlation fading model [19, Sect. II-B], the pdf of γ is approximately given by

$$p_\gamma(\gamma) = \frac{\gamma^{\frac{mL_r^2}{r}-1} \exp\left(-\frac{mL_r\gamma}{r\bar{\gamma}}\right)}{\Gamma\left(\frac{mL_r^2}{r}\right)\left(\frac{r\bar{\gamma}}{mL_r}\right)^{\frac{mL_r^2}{r}}}; \quad \gamma \geq 0 \tag{37}$$

where

$$r = L_r + \frac{2\sqrt{\rho}}{1-\sqrt{\rho}}\left(L_r - \frac{1-(\sqrt{\rho})^{L_r}}{1-\sqrt{\rho}}\right). \tag{38}$$

Rewriting the average BER of (22) as

$$\begin{aligned} P_b(E) &= \int_0^\infty Q(\sqrt{2g\gamma})p_\gamma(\gamma)\,d\gamma \\ &= \frac{1}{\pi}\int_0^{\pi/2}\int_0^\infty p_\gamma(\gamma)\exp\left(-\frac{g\gamma}{\sin^2\theta}\right)d\gamma\,d\theta \end{aligned} \tag{39}$$

then using either (36) or (37), the inner integral on γ can be computed in closed form, leaving a single finite integral on θ. In particular, defining

$$C(g; L_r, \bar{\gamma}, m, \rho; \theta) \triangleq \int_0^\infty p_\gamma(\gamma)\exp\left(-\frac{g\gamma}{\sin^2\theta}\right)d\gamma \tag{40}$$

then

$$P_b(E) = \frac{1}{\pi}\int_0^{\pi/2} C(g; L_r, \bar{\gamma}, m, \rho; \theta)\,d\theta. \tag{41}$$

The closed-form expression for $C(g; L_r, \bar{\gamma}, m, \rho; \theta)$ has been evaluated in [21] for both the equicorrelated and exponentially correlated fading models with the results

$$\begin{aligned} &C_{\text{equi}}(g; L_r, \bar{\gamma}, m, \rho; \theta) \\ &\quad = \left(1 + \frac{\bar{\gamma}g}{m\sin^2\theta}(1-\sqrt{\rho}+L_r\sqrt{\rho})\right)^{-m} \\ &\qquad \times \left(1 + \frac{\bar{\gamma}g}{m\sin^2\theta}(1-\sqrt{\rho})\right)^{-m(L_r-1)} \end{aligned} \tag{42a}$$

$$C_{\text{exp}}(g; L_r, \bar{\gamma}, m, \rho; \theta) = \left(1 + \frac{\bar{\gamma}gr}{mL_r\sin^2\theta}\right)^{-mL_r^2/r}. \tag{42b}$$

It should be noted that (41) together with (42a) is equivalent to [19, (32)], which is expressed in terms of the Appell hypergeometric function $F_2(\cdot;\cdot;\cdot;\cdot)$, which typically is not available in standard software libraries such as Mathematica, Matlab, or Maple and which is defined either in terms of an infinite range integral of a special function [19, (A-12)] or as a doubly infinite sum [19, (A-13)]. It should also be noted that (41) together with (42b) is equivalent to [19, (40)], which is expressed in terms of the Gauss hypergeometric function ${}_2F_1(\cdot,\cdot,\cdot;\cdot)$.

[8] It should be noted that in [19, (18)], the symbol ρ is used to denote the correlation coefficient of the underlying Gaussian processes that produce the fading on the channels. This correlation coefficient is equal to the square root of the power correlation coefficient, which for all practical purposes can be assumed to be equal to the envelope correlation coefficient. In this paper, we denote the envelope correlation coefficient by ρ so as to follow what seems to be the more conventional usage of this symbol.

$$p_\gamma(\gamma) = \frac{\left(\frac{\gamma m}{\bar{\gamma}}\right)^{L_r m-1}\exp\left(-\frac{\gamma m}{(1-\sqrt{\rho})\bar{\gamma}}\right){}_1F_1\left(m; L_r m; \frac{L_r m\sqrt{\rho}\gamma}{(1-\sqrt{\rho})(1-\sqrt{\rho}+L_r\sqrt{\rho})\bar{\gamma}}\right)}{\left(\frac{\bar{\gamma}}{m}\right)(1-\sqrt{\rho})^{m(L_r-1)}(1-\sqrt{\rho}+L_r\sqrt{\rho})^m\Gamma(L_r m)}, \qquad \gamma \geq 0 \tag{36}$$

C. Average SER for M-ary Signals

1) Multichannel MRC Reception of M-PSK: The SER for M-PSK over an AWGN is given by the integral expression [5, (71)], [4, (5)], [13, (3.119)]

$$P_s(E) = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left(-\frac{g_{\mathrm{PSK}}E_s}{N_0 \sin^2\theta}\right) d\theta \qquad (43)$$

where $g_{\mathrm{PSK}} = \sin^2(\pi/M)$ and E_s/N_0 is the received symbol SNR. For MRC RAKE reception in the presence of the fading channel model of (15), the conditional SER is obtained from (43) by replacing E_s/N_0 by $\gamma_s \triangleq \alpha_{\mathrm{MRC}}E_s/N_0 = (\sum_{l=1}^{L_r}\alpha_l^2)E_s/N_0$ where γ_s represents the instantaneous SNR per symbol after combining. Following the same steps as in (23), it is straightforward to show that the SER over generalized fading channels is given by [14]

$$P_s(E) = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \prod_{l=1}^{L_r} I_l(\bar{\gamma}_l, g_{\mathrm{PSK}}, \theta)\, d\theta \qquad (44)$$

where $I_l(\bar{\gamma}_l, g_{\mathrm{PSK}}, \theta)$ is defined in (24), with γ_l now denoting the instantaneous *symbol* SNR for the lth path and $\bar{\gamma}_l$ the average *symbol* SNR for the same path. The expressions for $I_l(\bar{\gamma}_l, g_{\mathrm{PSK}}, \theta)$ for the various fading channel models have already been given in Table 3 and can be used in (44) to compute the average SER for M-PSK over generalized fading channels.

We conclude this section by noting that results for multichannel reception with EGC and those for correlated fading amplitudes can be obtained in a manner similar to the approaches in Section VI-B2b (see [16]) and Section VI-B2c (see [21]), respectively.

2) Multichannel MRC Reception of M-QAM: For a square M-QAM signal constellation with $M = 2^k$ points (k even), the conditional (on the fading) SER is obtained from the AWGN result [13, (10.32)] as

$$\begin{aligned} P_s(E;\gamma_s) &= 4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{2g_{\mathrm{QAM}}\gamma_s}\right) \\ &\quad - 4\left(1-\frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{2g_{\mathrm{QAM}}\gamma_s}\right) \end{aligned} \qquad (45)$$

where $g_{\mathrm{QAM}} = 3/[2(M-1)]$. Using (2) and also the new representation for the square of the Gaussian Q-function given in (6), the average SER can be written as

$$\begin{aligned} P_s(E) &= \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)\int_0^{\pi/2}\prod_{l=1}^{L_r} I_l(\bar{\gamma}_l, g_{\mathrm{QAM}}, \theta)\, d\theta \\ &\quad - \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)^2 \int_0^{\pi/4}\prod_{l=1}^{L_r} I_l(\bar{\gamma}_l, g_{\mathrm{QAM}}, \theta)\, d\theta \end{aligned} \qquad (46)$$

where again $I_l(\bar{\gamma}_l, g_{\mathrm{QAM}}, \theta)$ is defined in (24) and tabulated in Table 3. Again, the results for correlated fading amplitudes can be found in a manner similar to the approach in Section VI-B2c (see [21]).

A special case of interest is the average SER performance of M-QAM over frequency-flat channels, which can be obtained from (45) by setting $L_r = 1$. Using [15, (2.562.1)], the following new closed-form result can be obtained for a Rayleigh channel [14]:

$$\begin{aligned} P_s(E) &= 2\left(1-\frac{1}{\sqrt{M}}\right) \\ &\quad \times \left(1-\sqrt{\frac{g_{\mathrm{QAM}}\bar{\gamma}}{1+g_{\mathrm{QAM}}\bar{\gamma}}}\right) + \left(1-\frac{1}{\sqrt{M}}\right)^2 \\ &\quad \times \left[\frac{4}{\pi}\sqrt{\frac{g_{\mathrm{QAM}}\bar{\gamma}}{1+g_{\mathrm{QAM}}\bar{\gamma}}}\tan^{-1}\left(\sqrt{\frac{1+g_{\mathrm{QAM}}\bar{\gamma}}{g_{\mathrm{QAM}}\bar{\gamma}}}\right) - 1\right] \end{aligned} \qquad (47)$$

where $\bar{\gamma} = \Omega E_s/N_0$ is the average received SNR per symbol. Note that the result in (47) agrees with that obtained in [22, (44)] for the special case of $M = 16$.

Another, more general case of interest that leads to a closed-form result is the average SER performance of M-QAM over L_r dissimilar Rayleigh fading channels. Using a partial fraction expansion of the integrand in (46), then with the help of [15, (2.562.1)], it can be shown that [14]

$$\begin{aligned} P_s(E) &= 2\left(1-\frac{1}{\sqrt{M}}\right)\sum_{l=1}^{L_r}\rho_l\left(1-\sqrt{\frac{g_{\mathrm{QAM}}\bar{\gamma}}{1+g_{\mathrm{QAM}}\bar{\gamma}}}\right) \\ &\quad + \left(1-\frac{1}{\sqrt{M}}\right)^2 \\ &\quad \times \left[\frac{4}{\pi}\sum_{l=1}^{L_r}\rho_l\sqrt{\frac{g_{\mathrm{QAM}}\bar{\gamma}}{1+g_{\mathrm{QAM}}\bar{\gamma}}}\right. \\ &\quad \left. \times \tan^{-1}\left(\sqrt{\frac{1+g_{\mathrm{QAM}}\bar{\gamma}}{g_{\mathrm{QAM}}\bar{\gamma}}}\right) - \sum_{l=1}^{L_r}\rho_l\right] \end{aligned} \qquad (48)$$

where

$$\rho_l = \left(\prod_{\substack{k=1 \\ k\neq l}}^{L_r}\left(1-\frac{\bar{\gamma}_k}{\bar{\gamma}_l}\right)\right)^{-1}. \qquad (49)$$

Last, although not specifically treated here, the average SER performance of the one-dimensional case, M-AM, which is referenced in Table 1, can be derived in a manner similar to that presented in this section and is discussed in [14].

VII. Noncoherent and Differentially Coherent Multichannel Detection of Digital Signals

A. Average BER for Binary Signals

Many problems dealing with the BER performance of differentially coherent and noncoherent detection of PSK and FSK signals have a decision variable that is a quadratic form in independent complex-valued Gaussian random variables. Almost two decades ago, Proakis [23] developed a general expression for evaluating the probability of error for multichannel reception of such binary signals when the decision variable is in that particular form. Indeed, the development and results originally obtained in [23] later

Table 4 Special Cases of Multichannel Reception of Differentially Coherent and Noncoherent Detection of Digital Signals

Detection Type	Modulation (Signal Set)	Parameters of Marcum Q-Function
Noncoherent	Equal energy, equiprobable correlated binary signals (λ = complex correlation coefficient)	$\eta = 1, a = \sqrt{\frac{1-\sqrt{1-\lvert\lambda\rvert^2}}{2}}, b = \sqrt{\frac{1+\sqrt{1-\lvert\lambda\rvert^2}}{2}}$
	Equal energy, equiprobable uncorrelated binary signals, e.g., BFSK	$\eta = 1, a = 0, b = 1$
Differentially Coherent	Binary phase-shift-keying (DPSK)	$\eta = 1, a = 0, b = \sqrt{2}$
	Quadrature phase-shift-keying (DQPSK) with Gray coding	$\eta = 1, a = \sqrt{2-\sqrt{2}}, b = \sqrt{2+\sqrt{2}}$

appeared in [24, Appendix 4B] and have become a classic in the annals of communication system performance literature. The most general form of the bit error probability expression, i.e., [24, (4B.21)] obtained by Proakis, was given in terms of the first-order Marcum Q-function and modified Bessel functions of the first kind. Although implied but not explicitly given in [23] and [24], this general form can be rewritten in terms of the generalized Marcum Q-function of (13) as

$$\begin{aligned} P_b(E;\gamma) = Q_1(a\sqrt{\gamma}, b\sqrt{\gamma}) - \left[1 - \frac{\sum_{l=0}^{L_r-1}\binom{2L_r-1}{l}\eta^l}{(1+\eta)^{2L_r-1}}\right] \\ \times \exp\left[-\frac{(a^2+b^2)\gamma}{2}\right] I_0(ab\gamma) + \frac{1}{(1+\eta)^{2L_r-1}} \\ \times \left[\sum_{l=2}^{L_r}\binom{2L_r-1}{L_r-1}\eta^{L_r-l}\right. \\ \times [Q_l(a\sqrt{\gamma}, b\sqrt{\gamma}) - Q_1(a\sqrt{\gamma}, b\sqrt{\gamma})] \\ - \sum_{l=2}^{L_r}\binom{2L_r-1}{L_r-l}\eta^{L_r-1+l} \\ \left.\times [Q_l(b\sqrt{\gamma}, a\sqrt{\gamma}) - Q_1(b\sqrt{\gamma}, a\sqrt{\gamma})]\right] \end{aligned} \quad (50)$$

where $\gamma = \sum_{l=1}^{L_r}\gamma_l$ is the total instantaneous SNR per bit

$$a = \sqrt{\frac{2A_3}{\gamma}\frac{v_1/v_2}{v_1+v_2}}, \quad b = \sqrt{\frac{2A_2}{\gamma}\frac{v_2/v_1}{v_1+v_2}} \quad (51)$$

and $\eta = v_2/v_1$, where the parameters v_1, v_2, A_2, A_3 are defined in [24, (4B.6)] and [24, (4B.10)], respectively.

A number of special cases of (50) corresponding to specific modulation/detection schemes are of particular importance and are tabulated in Table 4. Note that in all cases (as previously alluded to in Section V-B), a and b (and hence their ratio) are independent of the fading channel model and hence can be treated as constants when averaging the conditional BER over γ. More about this shortly.

For $L_r = 1$ (i.e., single channel reception), the latter two summations in (50) do not contribute, and hence one immediately obtains the result in [24, (4B.21)], i.e.,

$$\begin{aligned} P_b(E;\gamma) = Q_1(a\sqrt{\gamma}, b\sqrt{\gamma}) - \left(\frac{\eta}{1+\eta}\right) \\ \times \exp\left[-\frac{(a^2+b^2)\gamma}{2}\right] I_0(ab\gamma) \end{aligned} \quad (52)$$

which for $\eta = 1, a = 0$ and $b = 1$ ($b = \sqrt{2}$) reduces to the well-known expressions for orthogonal BFSK (DPSK) as reported in [24, (4.3.19)] ([24, (4.2.117)]), namely

$$\begin{aligned} P_b(E;\gamma)\,|_{\mathrm{BFSK}} &= \frac{1}{2}\exp\left(-\frac{\gamma}{2}\right) \\ P_b(E;\gamma)\,|_{\mathrm{DPSK}} &= \frac{1}{2}\exp(-\gamma). \end{aligned} \quad (53)$$

For $\eta = 1$ and any $L_r \geq 1$, which corresponds to the case of multichannel detection of equal energy correlated binary signals, after some simplification (50) becomes [25]

$$\begin{aligned} P_b(E;\gamma) = \frac{1}{2} + \frac{1}{2^{2L_r-1}}\sum_{l=1}^{L_r}\binom{2L_r-1}{L_r-l} \\ \times [Q_l(a\sqrt{\gamma}, b\sqrt{\gamma}) - Q_l(b\sqrt{\gamma}, a\sqrt{\gamma})] \end{aligned} \quad (54)$$

Once again setting $a = 0$ and $b = 1$ ($b = \sqrt{2}$), then using the series form for the lth-order Marcum Q-function in [26, (9)] and the combinatorial identity $\sum_{l=1}^{L_r}\binom{2L_r-1}{L_r-l} = 2^{2(L_r-1)}$, (54) reduces to the well-known expressions for orthogonal BFSK (DPSK) as reported in [24, (4.4.13)], namely

$$P_b(E;\gamma) = \frac{1}{2^{2L_r-1}}e^{-g\gamma}\sum_{l=0}^{L_r-1}c_l(g\gamma)^l \quad (55)$$

where

$$c_l = \frac{1}{l!}e^{-g\gamma}\sum_{k=0}^{L_r-1-l}\binom{2L_r-1}{k} \quad (56)$$

and as before $g = 1/2$ for BFSK and $g = 1$ for DPSK.

To evaluate average BER in the same manner as was done for coherent reception, we will first need to substitute the alternate representations of the Marcum Q-function found in (9), (10), and (14) into the appropriate conditional BER expression, i.e., (50), (52), or (54). In the most general case, namely, (50), the result can be written as a single integral with finite limits and an integrand composed of elementary functions, i.e.,

$$\begin{aligned} P_b(E;\gamma) =& \frac{\eta^{L_r}}{2\pi(1+\eta)^{2L_r-1}} \\ &\times \int_{-\pi}^{\pi} \left[\frac{f(L_r;\zeta,\eta;\theta)}{1+2\zeta\sin\theta+\zeta^2} \right] \\ &\times \exp\left(-\frac{b^2\gamma}{2}\left[1+2\zeta\sin\theta+\zeta^2\right]\right) d\theta, \\ & 0^+ \leq \zeta = a/b < 1 \end{aligned} \quad (57)$$

where

$$f(L_r;\zeta,\eta;\theta) = f_0(L_r;\zeta,\eta;\theta) + f_1(L_r;\zeta,\eta;\theta) \quad (58)$$

with

$$\begin{aligned} f_0(L_r;\zeta,\eta;\theta) =& \left[-\frac{(1+\eta)^{2L_r-1}}{\eta^{L_r}} + \sum_{l=1}^{L_r} \binom{2L_r-1}{L_r-l} \right] \\ &\times \zeta(\zeta+\sin\theta) \end{aligned} \quad (59a)$$

$$\begin{aligned} f_1(L_r;\zeta,\eta;\theta) =& \sum_{l=1}^{L_r} \binom{2L_r-1}{L_r-l} \\ &\times \left[\left(\eta^{-l}\zeta^{-(l-1)} - \eta^{l-1}\zeta^{l+1}\right) \right. \\ &\times \cos((l-1)(\theta+\pi/2)) \\ &- \left(\eta^{-l}\zeta^{-(l-2)} - \eta^{l-1}\zeta^{l}\right) \\ &\left. \times \cos(l(\theta+\pi/2)) \right] \end{aligned} \quad (59b)$$

Note that in (57), the total instantaneous SNR per bit (over which we must average) appears only in the argument of the exponential term in the integrand. This is the identical behavior as was found for the analogous result corresponding to coherent reception. Also note that as $\zeta \to 0$, (57) assumes an indefinite form, and thus an analytical expression for the limit is more easily obtained from another form of the expression for the error probability, namely, (55) with g replaced by $b^2/2$. We further point out that the limit of (57) as $\zeta \to 0$ converges smoothly to the exact BER expression of (55). For example, numerical evaluation of (57) setting $\zeta = 10^{-3}$ $(a = 10^{-3}, b = 1)$ gives an accuracy of five digits when compared with numerical evaluation of (55) for the same system parameters. The representation (57) is therefore useful even in this specific case. This is particularly true for the performance of binary FSK and binary DPSK, which cannot be obtained via the classical representation of (55) in the most general fading case but which can be solved using (57). The results for the special cases of single channel reception $(L_p = 1)$ and $\eta = 1, L_r \geq 1$ can be easily obtained from (57) together with (58) and (59) and can be found in [25].

Consider the evaluation of the average BER for the case where the channel SNR's $\{\gamma_l; l = 1, 2, \cdots, L_p\}$ are statistically independent (but not necessarily identically distributed). Analogous to (23) and (24), we obtain from (57)

$$\begin{aligned} P_b(E) =& \frac{\eta^{L_r}}{2\pi(1+\eta)^{2L_r-1}} \\ &\times \int_{-\pi}^{\pi} \left[\frac{f(L_r;\zeta,\eta;\theta)}{1+2\zeta\sin\theta+\zeta^2} \right] \\ &\times \prod_{l=1}^{L_r} J_l(\bar{\gamma}_l; b,\zeta,\theta)\, d\theta \end{aligned} \quad (60)$$

where

$$\begin{aligned} &J_l(\bar{\gamma}_l; b,\zeta,\theta) \\ &\quad \triangleq \int_0^{\infty} \exp\left(-\frac{b^2\gamma_l}{2}[1+2\zeta\sin\theta+\zeta^2]\right) p_{\gamma_l}(\gamma_l)\, d\gamma_l. \end{aligned} \quad (61)$$

Comparing (61) with (24), we observe that the two integrals have identical form insofar as their dependence on γ_l is concerned. In fact, the specific results for $J_l(\bar{\gamma}_l; b,\zeta,\theta)$ corresponding to each fading case in Table 3 can be obtained by replacing $g/\sin^2\theta$ with $b^2(1+2\zeta\sin\theta+\zeta^2)/2$ in the expressions for $I_l(\bar{\gamma}_l, g, \theta)$. Last, if the fading is identically distributed with the same average SNR per bit $\bar{\gamma}_1$ for all channels, then (60) reduces to

$$\begin{aligned} P_b(E) =& \frac{\eta^{L_r}}{2\pi(1+\eta)^{2L_r-1}} \\ &\times \int_{-\pi}^{\pi} \left[\frac{f(L_r;\zeta,\eta;\theta)}{1+2\zeta\sin\theta+\zeta^2} \right] [J_1(\bar{\gamma}_1; b,\zeta,\theta)]^{L_r}\, d\theta. \end{aligned} \quad (62)$$

It should also be mentioned that the average BER can be obtained for the case of correlated Nakagami-m fading channels and is discussed in [21].

For single channel reception $(L_r = 1)$, the average BER of (62) simplifies to

$$\begin{aligned} P_b(E) =& \frac{1}{2\pi(1+\eta)} \\ &\times \int_{-\pi}^{\pi} \left[\frac{1-\eta\zeta^2+\zeta(1-\eta)\sin\theta}{1+2\zeta\sin\theta+\zeta^2} \right] \\ &\times J_1(\bar{\gamma}_1; b,\zeta,\theta)\, d\theta \end{aligned} \quad (63)$$

which for many fading channel models can be expressed in closed form. For example, for Rayleigh fading, the result is [25]

$$\begin{aligned} P_b(E) =& \\ & \frac{1}{2}\left[1 - \frac{(1-\zeta^2)b^2\bar{\gamma}_1}{2\sqrt{1+b^2\bar{\gamma}_1(1+\zeta^2)+\left(\frac{b^2\bar{\gamma}_1}{2}\right)^2(1-\zeta^2)^2}} \right], \\ & 0^+ \leq \zeta = a/b < 1 \end{aligned} \quad (64)$$

which for the special case $a = 0$ and $b = 1$ ($b = \sqrt{2}$) agrees with the expressions reported by Proakis [24, (7.3.12)] ([24, (7.3.10)]) for orthogonal BFSK (DPSK). Also, for 4-DPSK where $a = \sqrt{2 - \sqrt{2}}$, $b = \sqrt{2 + \sqrt{2}}$, (64) agrees with a closed-form result obtained by Tjhung *et al.* [27, (18)] in a different form.

B. Average SER for M-ary Signals—Single Channel Detection of Classical M-DPSK

The SER for M-DPSK over an AWGN is given by the integral expression [5, (44)], [13, (7.7)]

$$\begin{aligned} P_s(E) &= \frac{\sin \frac{\pi}{M}}{2\pi} \\ &\quad \times \int_{-\pi/2}^{\pi/2} \frac{\exp\left(-\frac{E_s}{N_0}\left[1 - \cos\frac{\pi}{M}\cos\theta\right]\right)}{1 - \cos\frac{\pi}{M}\cos\theta}\, d\theta \\ &= \frac{\sqrt{g_{\text{PSK}}}}{2\pi} \\ &\quad \times \int_{-\pi/2}^{\pi/2} \frac{\exp\left(-\frac{E_s}{N_0}\left[1 - \sqrt{1 - g_{\text{PSK}}}\cos\theta\right]\right)}{1 - \sqrt{1 - g_{\text{PSK}}}\cos\theta}\, d\theta. \end{aligned} \tag{65}$$

For single channel detection in the presence of the multipath fading channel model, the conditional SER is obtained from (65) by replacing E_s/N_0 by $\gamma \triangleq \alpha^2 E_s/N_0$. When this is done, (65) will already be in the preferred form, namely, a single integral with finite limits and an integrand that is exponential (Gaussian) in the fading RV. By analogy with the results in Section VI-C1, it is straightforward to show that the SER over generalized fading channels is given by

$$\begin{aligned} &P_s(E) \\ &\quad = \frac{\sqrt{g_{\text{PSK}}}}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{1 - \sqrt{1 - g_{\text{PSK}}}\cos\theta} K(\bar{\gamma}, g_{\text{PSK}}, \theta)\, d\theta \end{aligned} \tag{66}$$

where

$$\begin{aligned} K(\bar{\gamma}, g_{\text{PSK}}, \theta) &\triangleq \int_0^\infty \exp\left(-\frac{\alpha^2 E_s}{N_0}\left[1 - \sqrt{1 - g_{\text{PSK}}}\cos\theta\right]\right) \\ &\qquad \times p_\alpha(\alpha)\, d\alpha \\ &= \int_0^\infty \exp(-\gamma[1 - \sqrt{1 - g_{\text{PSK}}}\cos\theta]) \\ &\qquad \times p_\gamma(\gamma)\, d\gamma. \end{aligned} \tag{67}$$

Once again, specific results for $K(\bar{\gamma}, g_{\text{PSK}}, \theta)$ corresponding to each fading case in Table 3 can be obtained by replacing $g/\sin^2\theta$ with $1 - \sqrt{1 - g_{\text{PSK}}}\cos\theta$ in the expressions for $I_l(\bar{\gamma}_l, g, \theta)$.

Using the SER results for the AWGN presented in [28], which are expressed in terms of the first-order Marcum Q-function, the average SER performance of multiple-symbol M-DPSK on a generalized fading channel can be evaluated in a manner similar to that discussed in Section VII-A. The details are omitted here for the sake of brevity.

VIII. Applications

Coupled with what already appears to be an overwhelming number of theoretical results are many practical applications that demonstrate that the unified approach has far more than academic value. We briefly mention some of these here, keeping in mind that a complete detailed treatment of each would require documentation in an equal number of journal articles.

We have already mentioned in Table 2 the environments that are characterized by the various fading channel models. Thus, it goes without saying that the unified approach allows simple evaluation of the BER performance of a wide class of satellite, terrestrial, and maritime mobile communication systems.

In association with the IS-95 standard for wireless communication, a great deal of interest has focussed in recent years on the use of direct sequence spread-spectrum modulation as a multiple access scheme (DS-CDMA) [29], [30]. While the initial contributions considered single carrier DS-CDMA, more recently, attention has turned to multicarrier DS-CDMA [31], which itself is a derivative of orthogonal frequency division multiplexing [32], [33]. Since in these techniques the self-interference induced by the autocorrelation of the users' spreading codes and the multiple access interference induced by the other users are typically modeled as additional Gaussian noise sources independent of the AWGN, then, treating the sum of these noise sources as a single equivalent WGN, the theoretical results presented in this paper can be applied to predict the additional BER degradation of these systems caused by the fading channel [34], [35].

As a means of obtaining additional diversity gain against the fading environment, a combination of space (multiple antennas) and path (MRC RAKE) diversity can be employed [21]. The BER performance of such two-dimensional diversity systems can be obtained as a straightforward extension of the theoretical results given in this paper for path diversity alone.

Last, there is a strong analogy between the conditional error-rate performance for diversity reception of an i.i.d. L_r-path received signal and the pair-wise error probability of two sequences (length L_r) of i.i.d. faded symbols, which is characteristic of error correction coded (e.g., convolutional, trellis) communication over a fading channel. In particular, the conditional BER of (17) together with the MRC sum of (18) also characterizes the above conditional pair-wise error probability with *known* channel state information. Similarly, (17) together with the EGC sum of (19) also characterizes the above conditional pair-wise error probability with *unknown* channel state information. As an example of how the unified approach benefits the evaluation of average BER in error correction coded systems, consider the transmission of trellis-coded M-PSK over a memoryless (independent fading from transmission to transmission) channel with known channel state information. In [36], the BER was derived for such a system in the form of a union-Chernoff bound, where the Chernoff bound portion applied to the pair-wise error probability and the union bound portion converted the pair-wise error probability to average BER using the transfer function bound method. Using now the alternate form of the Gaussian Q-function of (2) in (17) together with (18) and performing the average over the i.i.d. fading sequence enables one exactly to evaluate the pair-wise error probability, thus eliminating the need for the Chernoff bound. Hence, the resulting form for the average BER is strictly a union (as opposed to a union-Chernoff) bound and as such is a tighter bound to the true result. The full details of this approach are given in [6].

IX. Conclusion

We have shown that by employing alternate forms of the Gaussian and Marcum Q-functions, it is possible to unify the error-probability performance of coherent, differentially coherent, and noncoherent communications in the presence of generalized fading under a single common framework where the results are, with little exception, expressible in a form that lends itself to simple evaluation and furthermore provides additional insight into the dependence of this performance on the system parameters. While we have already exploited many of the potential applications of the unified approach presented here and plan to continue to do so in the future, we also hope that this paper will serve as an inspiration to other researchers to do the same. We fully hope that the words of one of the reviewers, who stated that "the paper will have a long and useful reference life," will truly become a reality.

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