

1. BACKGROUND AND REVIEW

Some reminders before we start the course proper. Get out all those old notes, and come up to speed again...

1.1 1- and 2-Sided Z-Transforms

→ Read Haykin Sections 1.1 to 1.3 and Appendix A

- Many of the signals we'll deal with are non-causal; that is, they are non zero for negative time. Need tools to analyze them.

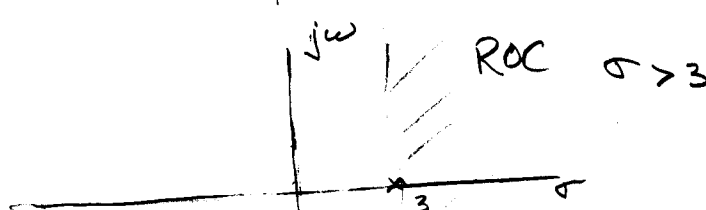
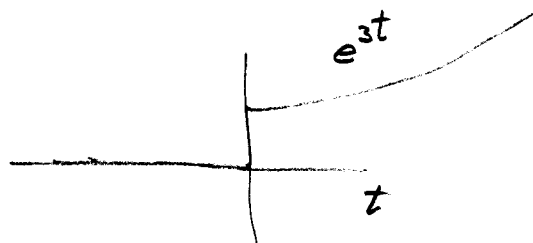
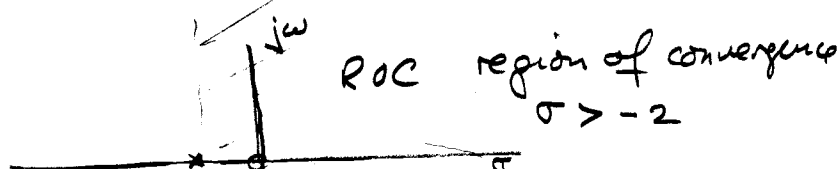
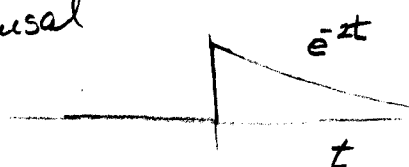
Read "One Sided and Two Sided Laplace Transforms" at web site

- Recall Laplace 2 sided

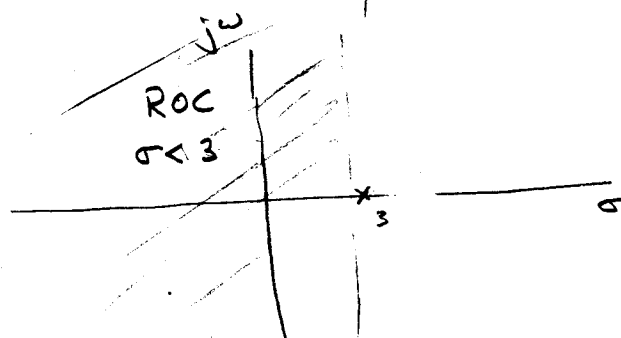
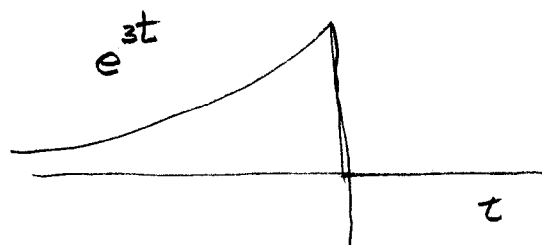
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

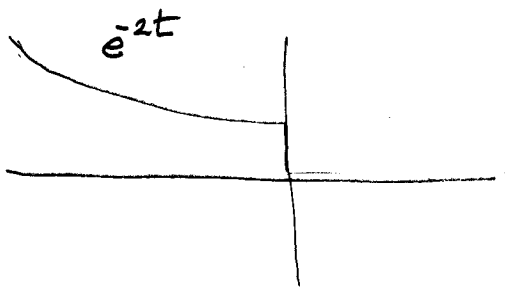
where $\sigma = \text{Re}[s]$ is chosen for convergence of integral

causal

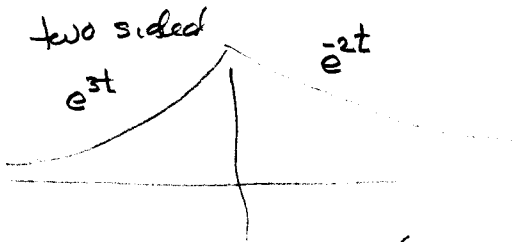
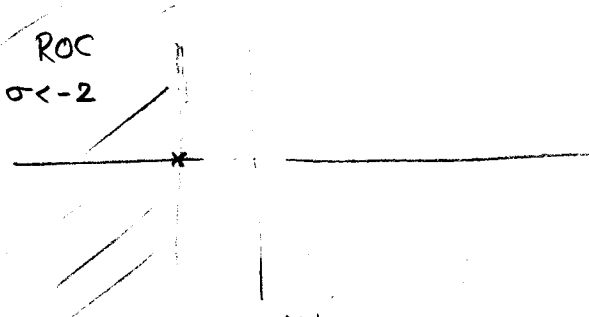


anticausal



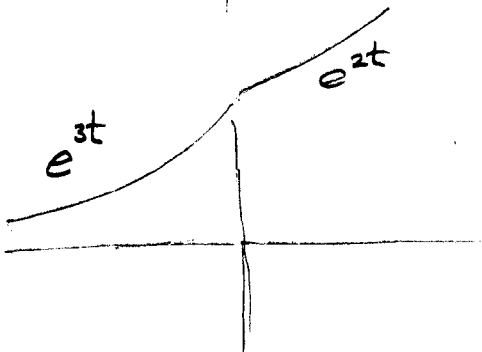


ROC
 $\sigma < -2$



ROC
 $-2 < \sigma < 3$

$\sigma > -2$
and $\sigma < 3$

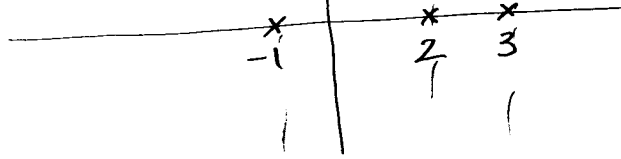


ROC
 $\sigma > 2$
and $\sigma < 3$

Observations:

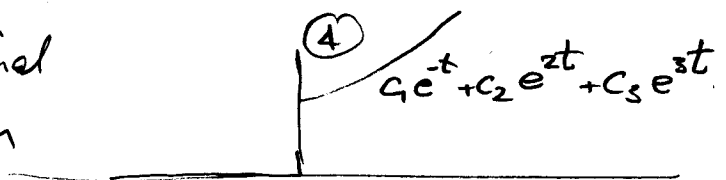
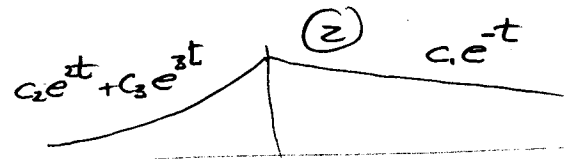
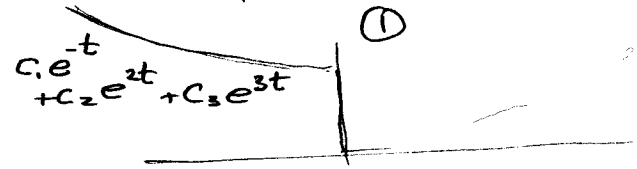
- different signals can produce same transform, so need to specify ROC
- positive time produces poles to left of ROC; negative time to the right

① | ② | ③ | ④

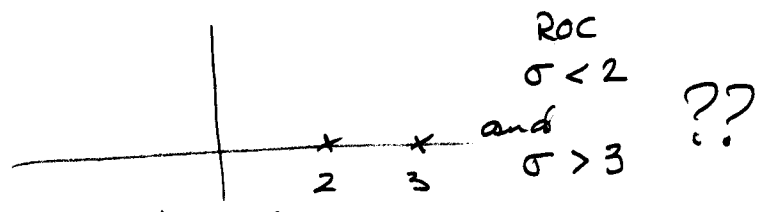
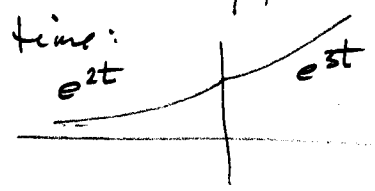


one transform, choice of 4 ROCs

- don't always have poles (rational transform), but if we do, then can use residues



occasionally, don't get a single ROC for both pos, neg time:



In this case, carry around two transforms

$$X_-(s) = -\frac{1}{s-2}, \sigma < 2$$

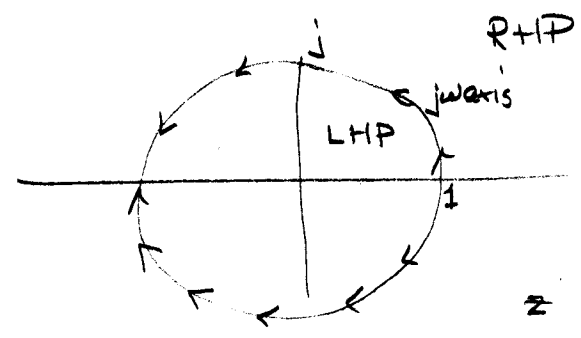
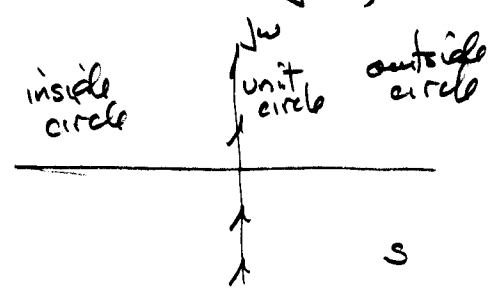
and invert the appropriate one

$$X_+(s) = \frac{1}{s+3}, \sigma > 3$$

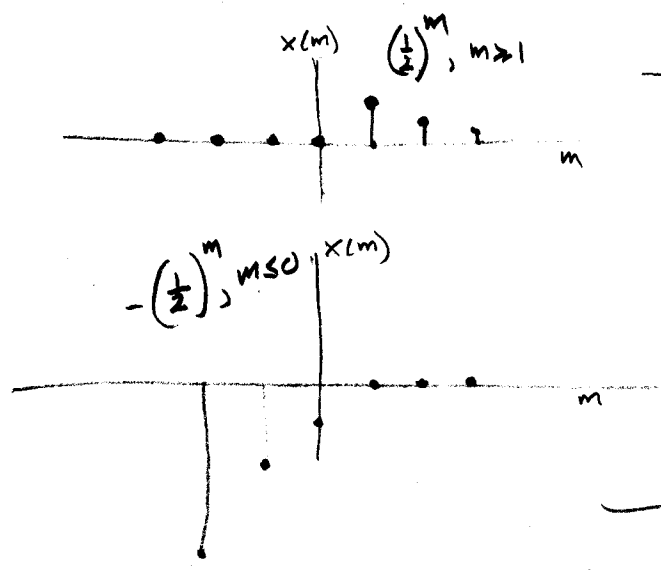
• Similar pattern for discrete time, two sided.

$$\sum_{m=-\infty}^{\infty} x(m) e^{-smT} = \sum_m x(m) (e^{sT})^{-m} = \sum_m x(m) z^{-m}$$

since $s = \sigma + j\omega$, $z = e^{\sigma T} e^{j\omega T}$

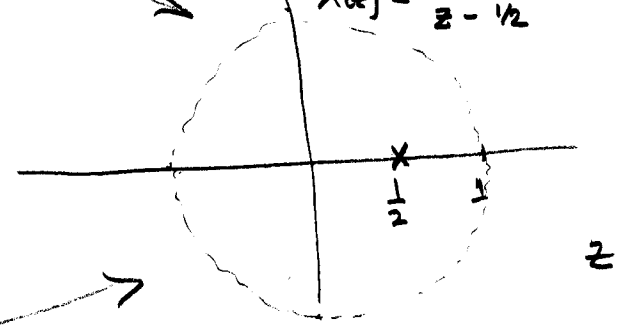


$X(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-m}$ with $|z|$ large enough for convergence $m > 0$
small enough $\dots \dots \dots m < 0$



ROC $|z| > \frac{1}{2}$

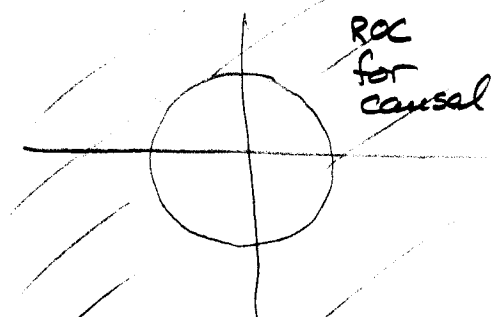
$$X(z) = \frac{1/2}{z - 1/2}$$



ROC $|z| < \frac{1}{2}$

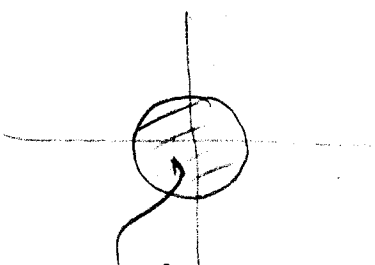
two different sequences, same transform

So if there is a single ROC, it might be annular



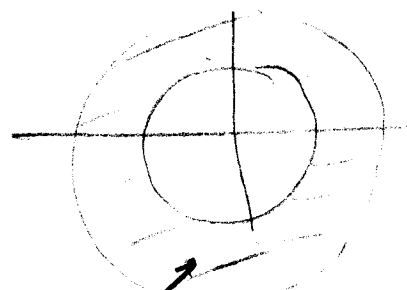
ROC
for
causal

if causal, all poles
inside ROC; inside
unit circle if stable



ROC for anticausal

if anticausal, all poles
outside ROC; outside
unit circle if stable



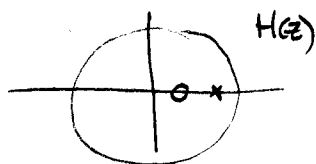
ROC for two
sided

poles inside and
outside ROC; if stable
for both $t < 0$, $t > 0$ then
ROC contains unit circle.

inverse filters?

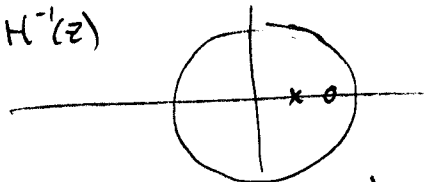
consider causal $h(k)$, $k \geq 0 \iff H(z) = \frac{\prod_i (z - z_i)}{\prod_k (z - p_k)}$

if $H(z)$ has all poles and zeroes in
the unit circle, it's stable, and there's
a causal, stable inverse filter

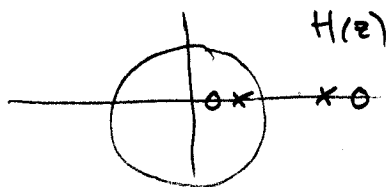


$H(z)$

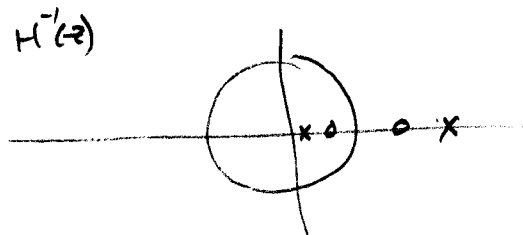
$$G(z) = H^{-1}(z)$$



consider stable, non causal — ROC contains unit circle
it has stable, non causal inverse



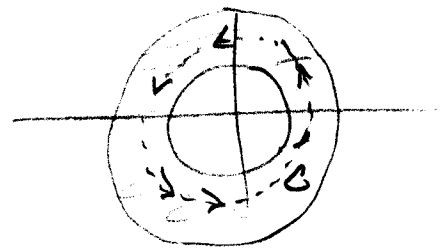
$H(z)$



$H^{-1}(z)$

• inversion of z transforms (general expression):

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^n \frac{dz}{z}; \quad C \text{ is in ROC}$$



for circular C in ROC: $z = Re^{j\theta}$ $dz = j d\theta Re^{j\theta}$

so $\frac{dz}{z} = j d\theta$

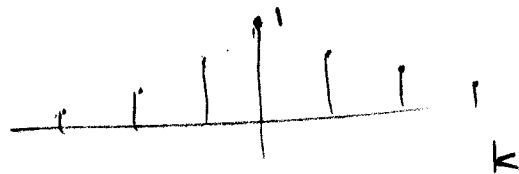
and $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(Re^{j\theta}) R^n e^{jn\theta} d\theta$

and if $R=1$, $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) e^{jn\theta} d\theta \leftarrow \text{Fourier series}$

• in special case of rational $X(z)$:

$$x(n) = \begin{cases} \sum_{\substack{p_i \text{ inside} \\ \text{ROC}}} \text{Res}_i [X(z) z^{n-1}] & n \geq 0 \\ - \sum_{\substack{p_i \text{ outside} \\ \text{ROC}}} \text{Res}_i [X(z) z^{n-1}] & n < 0 \end{cases} \quad \{p_i\} \text{ are poles}$$

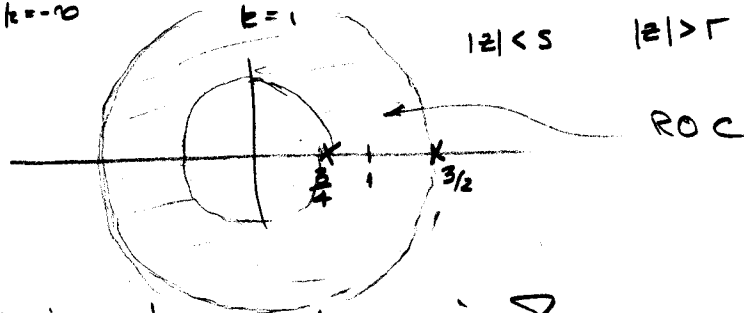
• example: $x(k) = \begin{cases} \left(\frac{3}{2}\right)^k & k \leq 0 \\ \left(\frac{3}{4}\right)^k & k > 0 \end{cases}$



$$\begin{aligned} X(z) &= \sum_{k=-\infty}^{\infty} x(k) z^{-k} = \sum_{k=-\infty}^0 \left(\frac{3}{2}\right)^k z^{-k} + \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k z^{-k} \\ &= \sum_{k=-\infty}^0 s^k z^{-k} + \sum_{k=1}^{\infty} r^k z^{-k} = \frac{s}{s-z} + \frac{r}{z-r} = \frac{z(r-s)}{(z-r)(z-s)} \end{aligned}$$

denote $s = 3/2$
 $r = 3/4$

$|z| < s \quad |z| > r$



Will inversion get us back again?

$k \geq 0$: $\text{Res}_{\frac{3}{4}} [X(z) z^{k-1}] = \left. \frac{z(r-s) z^{k-1}}{(z-s)} \right|_{z=r} = r^k$ ✓

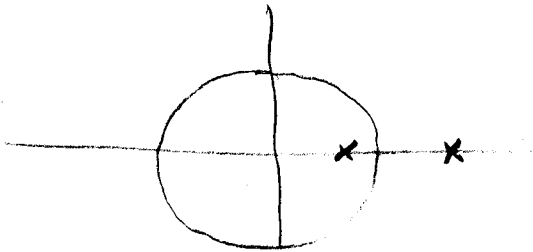
$k \leq 0$: $-\text{Res}_s [X(z) z^{k-1}] = - \left. \frac{z(r-s) z^{k-1}}{(z-r)} \right|_{z=s} = s^k$ ✓

• example: symmetric $x(n)$



poles are reciprocals

from previous page $X(z) = -\frac{5}{6} \frac{z}{(z - \frac{3}{2})(z - \frac{2}{3})}$



• some symmetry properties of $\mathcal{Z}[x(n)]$

- reversal: $\tilde{x}(n) = x(-n)$

$$\tilde{X}(z) = \mathcal{Z}[\tilde{x}(n)] = \sum_n x(-n) z^{-n} = \sum_k x(k) z^k = X(z^{-1})$$

compare with Fourier:

$$x(t) \leftrightarrow X(f) \quad x(-t) \leftrightarrow X(-f)$$

- conjugation $\mathcal{Z}[x^*(n)] = \sum_n x^*(n) z^{-n} = \left(\sum_n x(n) z^{*-n} \right)^* = X^*(z^*)$

again Fourier:

$$x^*(t) \leftrightarrow X^*(-f)$$

- conjugate reversal $x^*(-n)$

$$\mathcal{Z}[x^*(-n)] = \sum_n x^*(-n) z^{-n} = \left(\sum_k x(k) z^{*k} \right)^* = X^*(1/z^*)$$

- conjugate symmetry: if $x(-n) = x^*(n)$, that is $x^*(-n) = x(n)$

$$\text{then } X(z) = X^*(1/z^*)$$

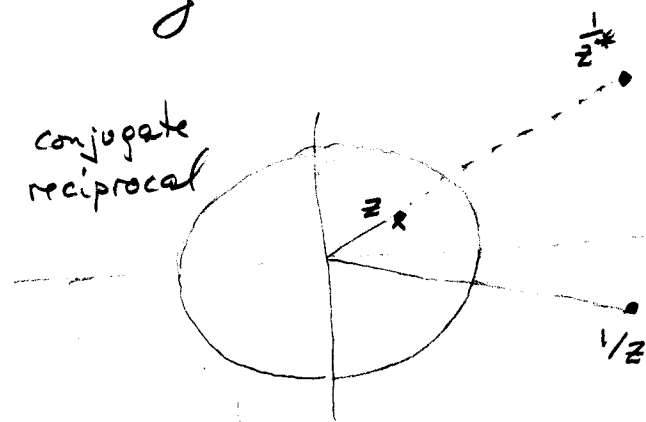
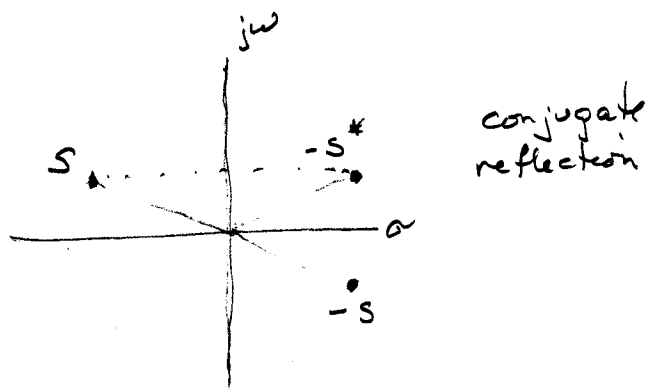
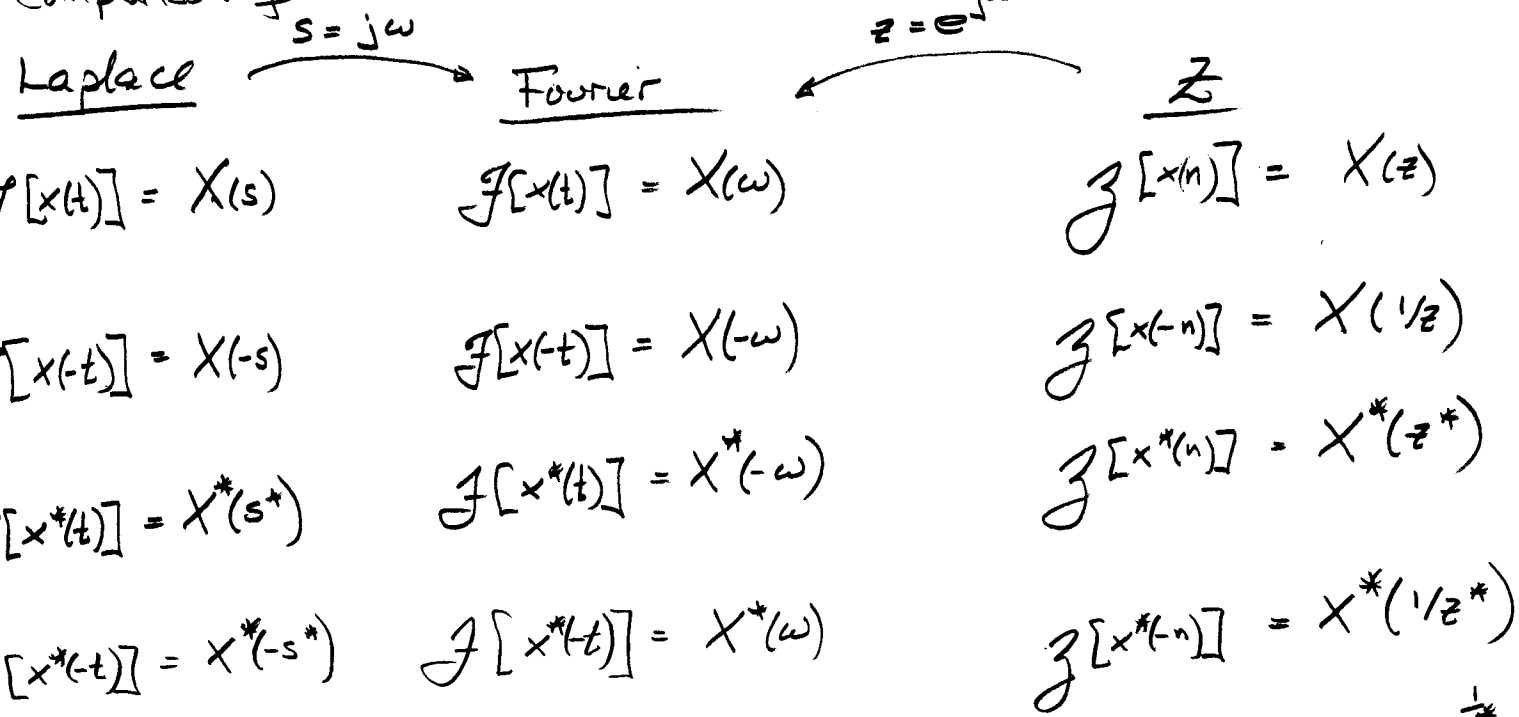
on the unit circle, where it is a Fourier transform,
 $z = e^{j\omega T} \quad 1/z^* = z = e^{j\omega T}$

$$X(\omega) = X^*(\omega) \Rightarrow X(\omega) \text{ is real, } X(z)|_{z=e^{j\omega T}} \text{ is real}$$

we know this already from Fourier

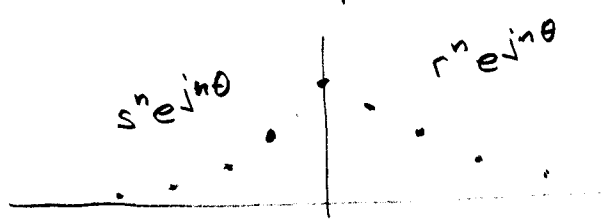
$$x^*(-t) = x(t) \Rightarrow X(f) \text{ is real}$$

• Comparison of transforms and reversals:



• example - check the expression for transform of conjugate reversal of $x(n)$.

- ① calculate $X(z)$
- ② calculate $\mathcal{Z}[x^*(-n)]$
- ③ see if it's $X^*(1/z^*)$



$$\begin{aligned}
 \textcircled{1} \quad X(z) &= \sum_{n=-\infty}^0 s^n e^{jn\theta} z^{-n} + \sum_{n=1}^{\infty} r^n e^{jn\theta} z^{-n} \\
 &= \sum_{n=0}^{\infty} \left(\frac{z}{s e^{j\theta}} \right)^n + \frac{r e^{j\theta}}{z} \frac{1}{1 - \frac{r e^{j\theta}}{z}} \\
 &= \frac{1}{1 - \frac{z}{s e^{j\theta}}} + \frac{r e^{j\theta}}{z - r e^{j\theta}} = \frac{s e^{j\theta}}{s e^{j\theta} - z} + \frac{r e^{j\theta}}{z - r e^{j\theta}}
 \end{aligned}$$

$$\textcircled{2} \left(\frac{1}{r} \right)^n e^{jn\theta} = \left(\frac{1}{r} \right)^n e^{jn\theta} \leftarrow x^*(-n)$$

$$\begin{aligned} \mathcal{Z}[x^*(-n)] &= \sum_{n=-\infty}^{-1} \left(\frac{e^{j\theta}}{r^2} \right)^n + \sum_{n=0}^{\infty} \left(\frac{e^{j\theta}}{s^2} \right)^n = \frac{r^2}{e^{j\theta}} \frac{1}{1 - \frac{r^2}{e^{j\theta}}} + \frac{s^2}{s^2 - e^{j\theta}} \\ &= \frac{r^2}{e^{j\theta} - r^2} + \frac{s^2}{s^2 - e^{j\theta}} \end{aligned}$$

$$\begin{aligned} \textcircled{3} X^*\left(\frac{1}{z^*}\right) &= \left[\frac{s e^{j\theta}}{s e^{j\theta} - \frac{1}{z^*}} + \frac{r e^{j\theta}}{\frac{1}{z^*} - r e^{j\theta}} \right]^* = \frac{s e^{-j\theta}}{s e^{-j\theta} - \frac{1}{z}} + \frac{r e^{-j\theta}}{\frac{1}{z} - r e^{-j\theta}} \\ &= \frac{s^2}{s^2 - e^{j\theta}} + \frac{r^2}{e^{j\theta} - r^2} \quad \checkmark \end{aligned}$$

• Questions for you:

- what happens to the ROC when we (a) conjugate (b) time-reverse (c) conjugate reverse a waveform (do Laplace and $\mathcal{Z}$)
- when you convolve two waveforms or sequences, what is the ROC of the result? what if the two ROCs do not overlap?

- One reason why we're interested in conjugate reversals:

$$x(t) \otimes x^*(-t) = \int x(t-\alpha) x^*(-\alpha) d\alpha = \int x(t+\beta) x^*(\beta) d\beta = \int x(\gamma) x^*(\gamma-t) d\gamma$$

= pulse autocorrelation function $R_x(t)$

$$x(n) \otimes x^*(-n) = \sum_k x(k) x^*(k-n) = \text{sequence autocorr.}$$

- What is the transform of the result?

- Use multiplication-convolution

$$\rightarrow \mathcal{L}[R_x(t)] = \mathcal{L}\left[\int x(\gamma) x^*(\gamma-t) d\gamma\right] = X(s) X^*(-s^*) \quad \left[\begin{array}{l} \text{energy} \\ \text{spectra} \end{array} \right]$$

$$\rightarrow \mathcal{Z}[x(n) \otimes x^*(-n)] = X(z) X^*(1/z^*)$$

demonstration:

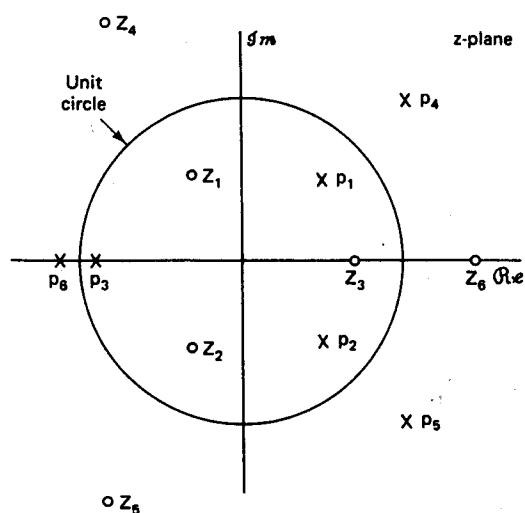
$$\begin{aligned} \mathcal{Z}\left[\sum_k x(k) x^*(k-n)\right] &= \sum_n \sum_k x(k) x^*(k-n) z^{-n} = \sum_i \sum_k x(k) x^*(i) z^{-(k-i)} \\ &= \left(\sum_k x(k) z^{-k}\right) \left(\sum_i x^*(i) z^i\right) = X(z) X^*(1/z^*) \end{aligned}$$

- Looks more familiar as Fourier

$$S_x(s) = X(s) X^*(-s^*) \xrightarrow{s=j\omega} X(j\omega) X^*(j\omega) \quad \left[\begin{array}{l} z = e^{j\omega} \\ S_x(z) = X(z) X^*(1/z^*) \end{array} \right]$$

$$= |X(j\omega)|^2$$

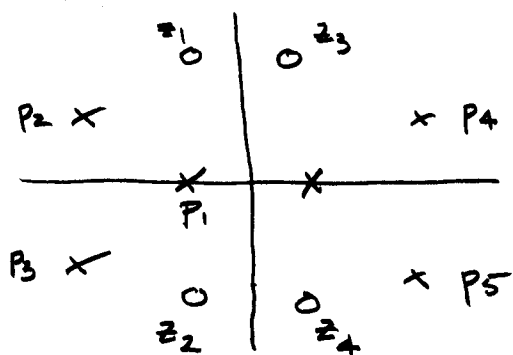
- Can we go back again? Given rational polynomial $|X(j\omega)|^2$, can we get $X(s)$ or $X(z)$, hence $x(t)$, $x(n)$?



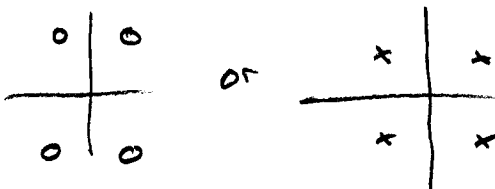
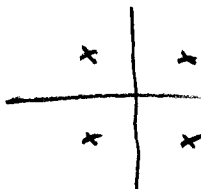
← poles and zeroes of $S_x(z)$.
roots linked through $z \leftrightarrow 1/z^*$

Similarly, for Laplace, $S_x(s)$

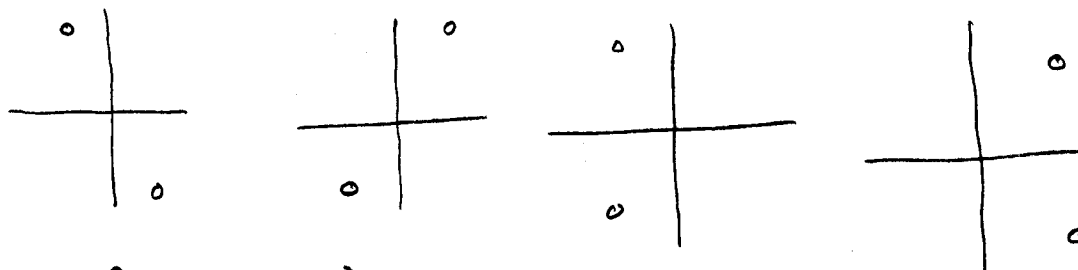
1.10



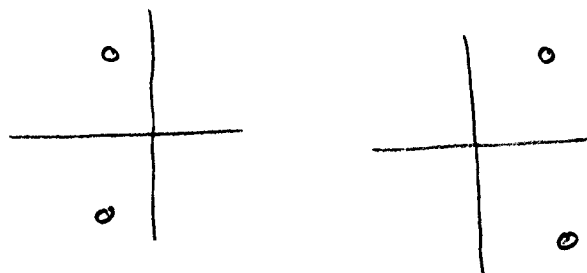
linked through
 $s \leftrightarrow -s^*$

• Consider one group  or 

- $X(s)$ could have 4 possibilities



- but for real impulse responses, it's only 2:

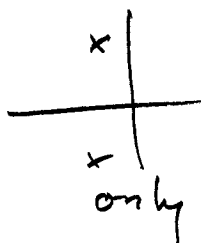


so they're in conjugate pairs

Thus order n (n poles, up to n zeroes) filter $X(s)$ has

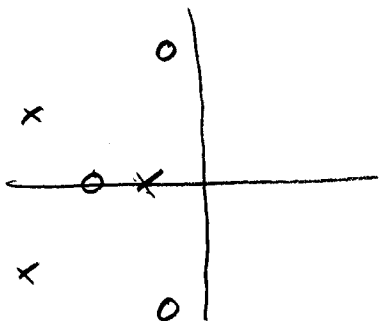
$$\underbrace{2^{n/2}}_{\text{poles}} \times \underbrace{2^{n/2}}_{\text{zeroes}} = 2^n \text{ possibilities}$$

- further restrict to stable $X(s)$, so poles in LHP

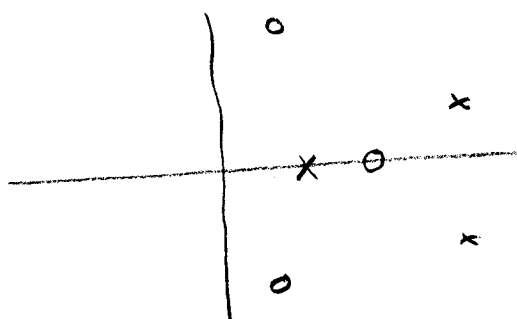


then $2^{n/2}$ possibilities, all associated with zeroes.

- and if all zeroes in LHP, we have stable and invertible $X(s)$ - unique "spectrum factorization"



$X(s)$



$X(-s^*)$

Such a filter - all poles, zeroes in LHP (unit circle) is termed "minimum phase" Oppenheim & Schaffer 5.4, 5.5, 5.6

- minimum group delay
- energy in $x(t)$ concentrated in earliest part

