1.4 Miscellaneous Covariance and Correlation Matrix Properties

$\boxed{\text{Haykin 4.2}}$

- Although Haykin puts them as properties of correlation matrices, most apply to covariance matrices

- Eigenvals $\{\lambda_i\}$ of R are real (cov matrix)

  positive semidefinite $\lambda_i \geq 0$

  neg semidef $\lambda_i \leq 0$

- Eigenvecs of distinct eigenvals are orthogonal: (cov)

  $$\lambda_i \neq \lambda_k \Rightarrow q_i^\dagger q_k = 0$$

- Eigenvecs of repeated eivals are linearly independent. (cov) Therefore any LC of them is an eigenvector, and we can build an orthonormal basis of this subspace, each basis vector an eigenvector.

- So we can arrange all eigenvecs orthonormal (cov)

  $$q_i^\dagger q_k = \delta_{ik} \quad \text{irrespective of distinct eigenvals}$$

- Spectral decomposition: (cov)

  $$\Lambda = Q^\dagger R Q \quad \text{so} \quad R = Q \Lambda Q^\dagger = Q \begin{bmatrix} \lambda_1 q_1^\dagger \\ \vdots \\ \lambda_M q_M^\dagger \end{bmatrix}$$

  or $$R = \sum_{i=1}^{M} \lambda_i q_i q_i^\dagger = \sum_{i=1}^{M} \lambda_i P_i \quad \text{a sum of rank-one matrices}$$

  $$= \sum \lambda_i \,[\,]\,\overline{}$$

  Rank one because $P_i \underline{w} = \underline{0}$ for any vector orthogonal to q_i and the dimensionality of that space is $M-1$. Hence $M-1$ zero eigenvals.

  BTW $\Lambda = Q^\dagger R Q$ is a unitary similarity transform

- $\mathrm{tr}[R] = \sum_{i=1}^{M} \lambda_i$ true of any matrix
- $|R| = \det[R] = \prod_{i=1}^{M} \lambda_i$ true of any matrix
- max (min) Rayleigh quotient:

$$\max_{\underline{x} \neq \underline{0}} \frac{\underline{x}^+ R \underline{x}}{\underline{x}^+ \underline{x}} = ?$$

same as $\max \underline{x}^+ R \underline{x}$ with $\underline{x}^+ \underline{x} = \text{const}$

express in eigenvec coords $\underline{x} = Q \underline{w}$

$$\max_{\underline{w}} \frac{\underline{w}^+ Q^+ R Q \underline{w}}{\underline{w}^+ Q^+ Q \underline{w}} = \frac{\underline{w}^+ \Lambda \underline{w}}{\underline{w}^+ \underline{w}} = \frac{\sum_{i=1}^{M} \lambda_i w_i^2}{\sum_{i=1}^{M} w_i^2}$$

Since the $w_i^2 \geq 0$, define $v_i = \frac{w_i^2}{\sum_{i=1}^{M} w_i^2}$ as nonneg weights that sum to 1

$$\max_{\underline{v}} \sum_{i=1}^{M} \lambda_i v_i$$

Now obvious:

- maximize the quotient with $v_1 = 1$, others 0
  $\underline{x} \propto \underline{q}_1$, max val of ratio is max eigenval λ_1
- minimum value is λ_M, with $\underline{x} \propto \underline{q}_M$

- Now try something tougher:

$$\max \frac{\underline{x}^+ R \underline{x}}{\underline{x}^+ N \underline{x}}$$

typically $\frac{\text{Signal power}}{\text{Noise power}}$

R, N Hermitian pos def

Here, define $N^{\frac{1}{2}}\underline{x} = \underline{w}$ $\quad \underline{x} = N^{-1/2}\underline{w}$

Note N is Hermitian, so $N^{+} = N$, $\; {N^{\frac{1}{2}}}^{+} = N^{\frac{1}{2}}$

So max
$$\frac{\underline{w}^{+} N^{-1/2} R N^{-\frac{1}{2}} \underline{w}}{\underline{w}^{+}\underline{w}}$$

Looks like the first problem, so

max (min) value is max (min) eigenval of $N^{-\frac{1}{2}} R N^{-\frac{1}{2}}$

or $N^{-1}R$, since $|N^{-\frac{1}{2}}RN^{-\frac{1}{2}} - \lambda I| = |N^{\frac{1}{2}}(N^{-1}R - \lambda I)N^{-\frac{1}{2}}|$

$$= |N^{\frac{1}{2}}||N^{-1}R - \lambda I||N^{-\frac{1}{2}}| = |N^{-1}R - \lambda I|$$

and it occurs for eigenvec

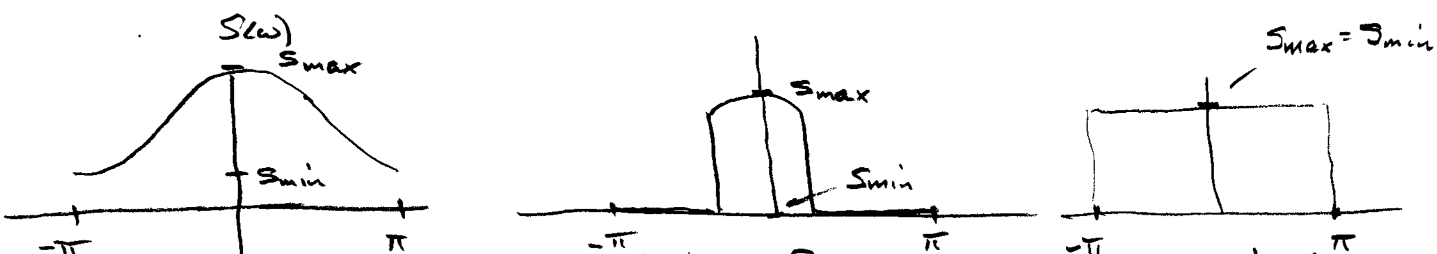
$$N^{1/2}(N^{-1}R - \lambda_0 I)\underbrace{N^{-1/2}\underline{\omega}_0}_{\underline{x}_0} = \underline{0}$$

so opt $\underline{x}$ is an eigenvector of $N^{-1}R$.

- For a set of samples $\underline{u}$ drawn from a process with power spectrum $S(\omega)$, the eigenvalues of the covariance matrix $R = \overline{\underline{u}\underline{u}^{+}}$ are bounded by

$$S_{min} = \min_{\omega} S(\omega) \le \lambda_i \le \max_{\omega} S(\omega) = S_{max}$$

and as M increases $\lambda_{min} \to S_{min}$ $\qquad \lambda_{max} \to S_{max}$



Caution: R probably singular

All eigenvals equal