

1.5 Signal Spaces [L.E. Franks Signal Theory, Prentice Hall '69] 5.1

- Many apparently different problems have the same form. This lets us learn a single set of principles, good for all. In particular, the connection between random variables and Euclidean spaces lets us use our spatial intuition as well as math.

- A vector space consists of a set of vectors, denoted $\underline{x}$ or $\underline{y}$ that forms a commutative group under a binary operator "+"

$$\underline{x} + \underline{y} = \underline{y} + \underline{x} \quad \underline{x} + (\underline{y} + \underline{z}) = (\underline{x} + \underline{y}) + \underline{z} \quad \underline{x} + \underline{0} = \underline{x} \quad \underline{x} + (-\underline{x}) = \underline{0}$$

with a field of scalars and scalar multiplication

$$\alpha(\beta \underline{x}) = (\alpha\beta) \underline{x} \quad 1 \underline{x} = \underline{x} \quad 0 \underline{x} = \underline{0} \quad \alpha(\underline{x} + \underline{y}) = \alpha \underline{x} + \alpha \underline{y}$$

$$(\alpha + \beta) \underline{x} = \alpha \underline{x} + \beta \underline{x}$$

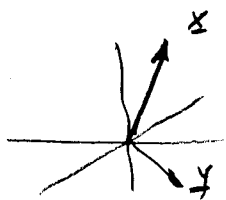
Usually the scalars are in $\mathbb{R}$ or $\mathbb{C}$, but $\mathbb{B} = \{0, 1\}$ will do, too

- An inner product space is a vector space with an inner product operation $(\mathcal{V} \times \mathcal{V}) \rightarrow \mathbb{C}$ denoted $(\underline{x}, \underline{y})$:

$$(\underline{x}, \underline{y}) = (\underline{y}, \underline{x})^* \quad (\alpha \underline{x} + \beta \underline{y}, \underline{z}) = \alpha (\underline{x}, \underline{z}) + \beta (\underline{y}, \underline{z})$$

$$(\underline{x}, \underline{x}) \geq 0, \text{ equality iff } \underline{x} = \underline{0}$$

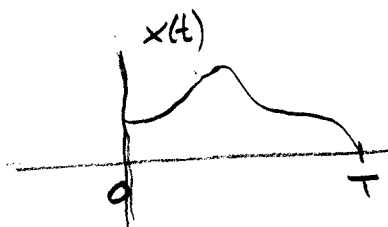
- Candidate vectors



little arrows in space

$$\underline{x} = \begin{bmatrix} 1 \\ 3-2j \\ \pi \\ j6 \end{bmatrix}$$

tuples of complex numbers (incl scalars)



functions in $L_2(T)$

$$\underline{x} \\ p_{\underline{x}}(\underline{x})$$

random variables

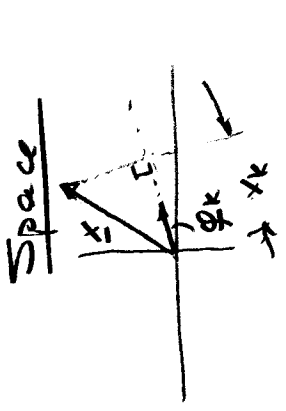
Concept	Denoted	$\mathbb{R}^n$	L_2	Random Vars	Space
scaling	$a\vec{x}$	$a\vec{x}$	$a \cdot x(t)$	$a \cdot x$	
sum	$\vec{x} + \vec{y}$	$\vec{x} + \vec{y}$	$x(t) + y(t)$	$x + y$	
inner prod	$(\vec{x}, \vec{y})$	$\vec{y}^T \vec{x}$	$\int_0^T x(t) y^*(t) dt$	$\sigma_{xy}^2 = E[xy^*]$	
orthogonality	$(\vec{x}, \vec{y}) = 0$	$\vec{y}^T \vec{x} = 0$	$\int_0^T x(t) y^*(t) dt = 0$	$\sigma_{xy}^2 = 0$	
norm	$\ \vec{x}\ $	$\sqrt{\vec{x}^T \vec{x}} = \vec{x} $	$\sqrt{\int_0^T x(t) ^2 dt} = \sqrt{E_x}$	σ_x	
distance	$d(\vec{x}, \vec{y}) = \ \vec{x} - \vec{y}\ $	$\sqrt{(\vec{x} - \vec{y})^T (\vec{x} - \vec{y})} = \sum_i x_i - y_i ^2$	$\sqrt{\int_0^T x(t) - y(t) ^2 dt}$	$z = x - y$ σ_z	
normalized inner prod	$\frac{(\vec{x}, \vec{y})}{\ \vec{x}\ \ \vec{y}\ }$	$\frac{\vec{y}^T \vec{x}}{\sqrt{\vec{x}^T \vec{x}} \sqrt{\vec{y}^T \vec{y}}}$	$\frac{\int_0^T x(t) y^*(t) dt}{\sqrt{\int_0^T x(t) ^2 dt} \sqrt{\int_0^T y(t) ^2 dt}}$	$\rho = \frac{\sigma_{xy}^2}{\sigma_x^2 \sigma_y^2}$	
triangle inequality	$\ \vec{x} + \vec{y}\ \leq \ \vec{x}\ + \ \vec{y}\ $	$ \vec{x} + \vec{y} \leq \vec{x} + \vec{y} $	$\sqrt{E_{x+y}} \leq \sqrt{E_x} + \sqrt{E_y}$	$\sigma_{x+y} \leq \sigma_x + \sigma_y$	

equality when $\vec{x} = a\vec{y}$

Concept
Component
projection
 Φ_k unit norm

Denoted Φ
 $x_k = (\underline{x}, \Phi)$
 $x_k = \Phi_k^+ x$

Random
 L_2
 $x_k = \int x(t) \Phi_k^+(t) dt$
 $x_k = \sigma_{x\Phi}^2$

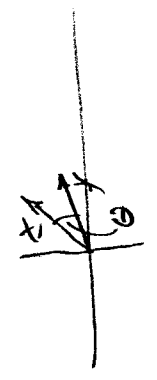


Schwarz
 $\neq ty$
equality when
 $\underline{x} = a \underline{y}$

$$|(\underline{x}, \underline{y})|^2 \leq \|\underline{x}\|^2 \|\underline{y}\|^2 \quad |\underline{y}^+ \underline{x}|^2 \leq \|\underline{y}\|^2 \|\underline{x}\|^2 \quad \left| \int x(t) y^*(t) dt \right|^2 \leq \int x(t) x^*(t) dt \int y(t) y^*(t) dt$$

$$|\sigma_{xy}^2| \leq \sigma_x^2 \sigma_y^2$$

$$\frac{2}{\|\underline{y}\|^2} |\cos \theta| \leq \frac{2}{\|\underline{x}\|^2} |\underline{y}|^2$$

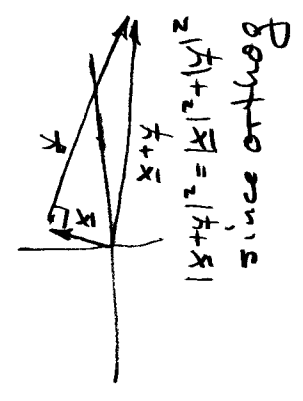


Pythagoras
for orthog

$$\|\underline{x} + \underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2 \quad \text{if } (\underline{x}, \underline{y}) = 0$$

$$\sum_{x+y} = \sum_x + \sum_y \quad \text{if } \sigma_{xy} = 0$$

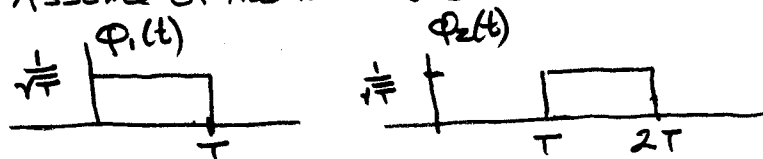
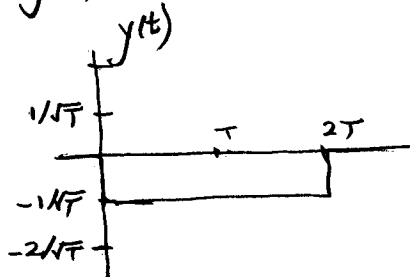
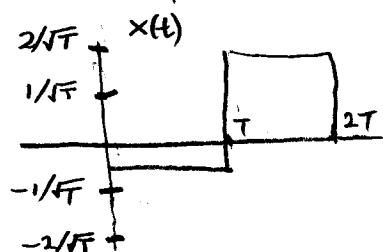
$$\sigma_{xy}^2 = \sigma_x^2 + \sigma_y^2 \quad \text{if } \sigma_{xy}^2 = 0$$



Pythagoras since orthog
 $\|\underline{x} + \underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2$

Example $L_2(T)$

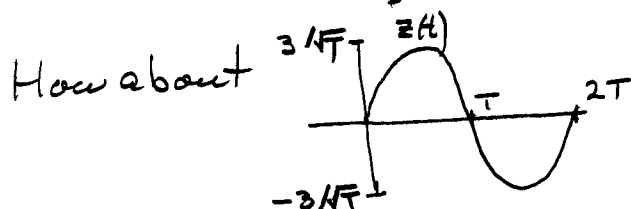
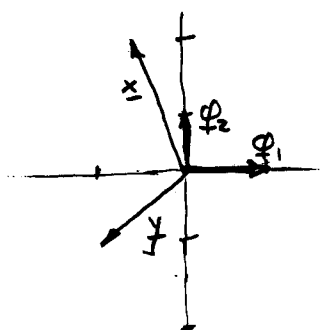
Assume orthonormal basis below:

To represent $x(t)$ and $y(t)$ below

$$x(t) = -0.5 \phi_1(t) + 2 \phi_2(t)$$

$$y(t) = -\phi_1(t) - \phi_2(t)$$

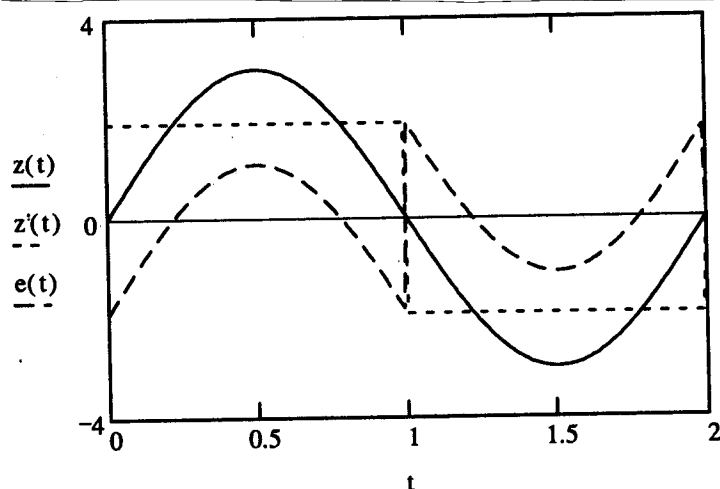
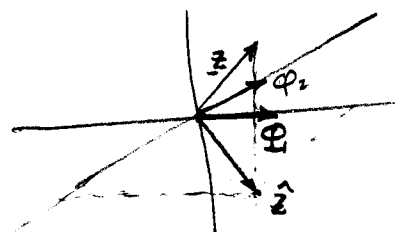
squared length is energy in waveform
e.g., energy in $x(t) - y(t)$ is $\|x - y\|^2$
 $= 3^2 + \frac{1}{2}^2 = 9.25$



energy is 9

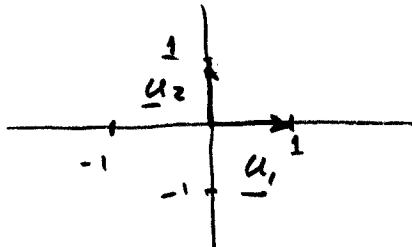
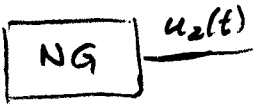
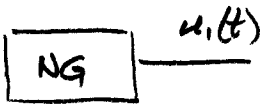
$$\begin{aligned} \hat{z}(t) &= (z, \phi_1) \phi_1(t) + (z, \phi_2) \phi_2(t) \\ &= \frac{6}{\pi} \phi_1(t) - \frac{6}{\pi} \phi_2(t) \end{aligned}$$

$$e(t) = z(t) - \hat{z}(t) \quad E_e = E_z - E_{\hat{z}} = 9 - \left(\frac{6}{\pi}\right)^2 - \left(\frac{6}{\pi}\right)^2$$

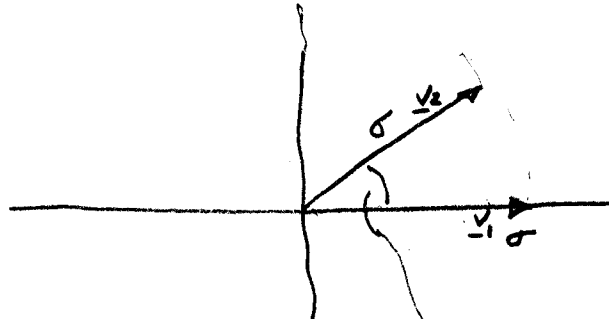


Example

Have two unit variance complex Gaussian noise generators, independent, and we want to make two correlated noises, corr'n coeff ρ , each variance σ^2 . How?



Have

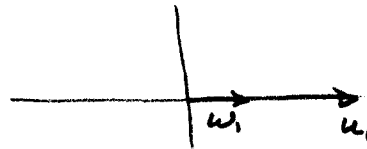


Want $\cos \theta = \rho$

Example GS again

Step 1 normalize u_1

$$w_1 = \frac{u_1}{\sigma_{u_1}}$$



Step 2 project u_2 onto w_1 space

$$\hat{u}_2 = (u_2, w_1) w_1 = \sigma_{u_2 w_1}^2 w_1$$

$e_2 = u_2 - \hat{u}_2$ has length σ_e

$$\sigma_e^2 = \sigma_{u_2}^2 - \sigma_{\hat{u}_2}^2$$

$$w_2 = \frac{e}{\sigma_e}$$

