# 4.3 Levinson's Recursion (Levinson–Durbin) $\boxed{\text{Haykin 6.3}}$

- Solution of $R^{-1}\underline{w}_f = \underline{r}$ requires $O(M^3)$ operations. The structure of the equations lends itself to recursive (or iterative) solutions — $O(M^2)$.

- First look at order recursion, with augmented forward W-H

$$\begin{bmatrix} r(0) & r(1) & \cdots & r(M-1) & r(M) \\ r(1)^* & \ddots & & r(M-2) & r(M-1) \\ & & \ddots & & \\ & & & r(1) & r(2) \\ r^*(M-1) & & r^*(1) & r(0) & r(1) \\ r^*(M) & r^*(M-1) & & r^*(1) & r(0) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -w_{f1} \\ \vdots \\ -w_{f,M-2} \\ -w_{f,M-1} \\ -w_{fM} \end{bmatrix} = \begin{bmatrix} P_M \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

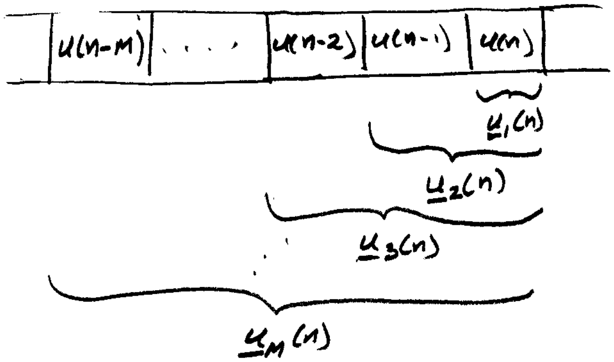
Each problem has the one of next lower order embedded in it. Same is true for backward equations.

- Next, consider relation between forward and backward partitionings of $R$ ($(M+1) \times (M+1)$)

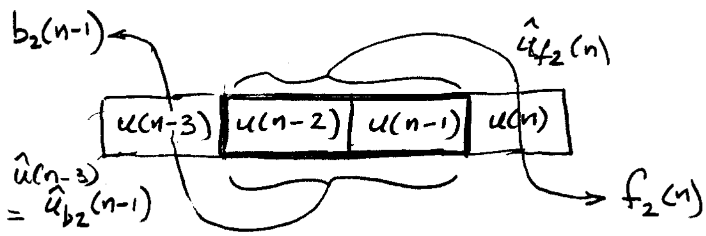
$$R_{M+1} = \begin{bmatrix} r(0) & r(1) & \cdots & r(M-2) & r(M-1) & r(M) \\ r^*(1) & r(0) & & & r(M-2) & r(M-1) \\ & & & & & r(M-2) \\ & & & & & \\ r^*(M-1) & & & & r(0) & r(1) \\ r^*(M) & r^*(M-1) & & & r^*(1) & r(0) \end{bmatrix} = \left[\begin{array}{c|c} r(0) & \underline{r}_M^{+} \\ \hline \underline{r}_M & R_M \end{array}\right] = \left[\begin{array}{c|c} R_M & \underline{\tilde{r}}_M^{*} \\ \hline \underline{\tilde{r}}_M^{T} & r(0) \end{array}\right]$$

- Can we construct order $m$ solution $\underline{a}_m$ by some calculations on $\underline{a}_{m-1}$?

Recall definitions




and now consider simultaneous forward and backward estimation (order 2 for simplicity)




both estimates based on $\underline{u}_2(n-1)$

The two prediction errors $b_2(n-1)$ and $f_2(n)$ are still correlated (unless $u(n)$ is an AR process of order 2 or less). Therefore we can estimate one in terms of the other.

First, estimate $f_2(n)$ with $b_2(n-1)$.

$$\hat{f}_2(n) = \gamma_2^* b_2(n-1) \qquad \gamma_2 = \frac{E[f_2^*(n)\, b_2(n-1)]}{E[|b_2(n-1)|^2]}$$

is the PARCOR coefficient

The error in this estimate is $f_3(n)$, since $b_2(n-1)$ is what's left in $u(n-3)$ after projection onto $\underline{u}_2(n-1)$. So

$$f_3(n) = f_2(n) - \gamma_2^* b_2(n-1)$$

$$\underline{a}_{f_3}^{+} \underline{u}_4(n) = [\underline{a}_{f_2}^{+} \mid 0]\, \underline{u}_4(n) - \gamma_2^* [0 \mid \underline{a}_{b2}^{+}]\, \underline{u}_4(n)$$

$$\underline{a}_{f_3} = \begin{bmatrix} \underline{a}_{f_2} \\ \hline 0 \end{bmatrix} - \gamma_2 \begin{bmatrix} 0 \\ \hline \tilde{\underline{a}}_{f_2}^{*} \end{bmatrix}$$

a recursion — we just need to find $\gamma_2$

since $\underline{a}_{b2} = \tilde{\underline{a}}_{f_2}^{*}$

We could also have estimated $b_2(n-1)$ with $f_2(n)$:

$$\hat{b}_2(n-1) = h_2^* f_2(n) \qquad h_2 = \frac{E[b_2^*(n-1) f_2(n)]}{E[|f_2(n)|^2]}$$

But $E[|f_2(n)|^2] = E[|b_2(n-1)|^2]$ (why?) so $h_2 = \gamma_2^*$ and

$$\hat{b}_2(n-1) = \gamma_2 f_2(n)$$

The error $b_2(n-1) - \hat{b}_2(n-1)$ is $b_3(n)$, since $\hat{b}_2(n-1) + \hat{u}_{b_2}(n-1)$ is effectively an estimate of $u(n-3)$ based on $u(n-2), u(n-1), u(n)$.

$$b_3(n) = b_2(n-1) - \gamma_2 f_2(n)$$

$$\underline{a}_{b3}^{\dagger} \underline{u}_4(n) = [0 \mid \underline{a}_{b2}^{\dagger}] \underline{u}_4(n) - \gamma_2 [\underline{a}_{f2}^{\dagger} \mid 0] \underline{u}_4(n)$$

$$\underline{a}_{b3} = \begin{bmatrix} 0 \\ \hline \underline{a}_{b2} \end{bmatrix} - \gamma_2^* \begin{bmatrix} \underline{a}_{f2} \\ \hline 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \hline \underline{a}_{b2} \end{bmatrix} - \gamma_2^* \begin{bmatrix} \tilde{\underline{a}}_{b2}^* \\ \hline 0 \end{bmatrix}$$

This is a recursion similar to the one for forward filters. In fact, they are the same: apply $\sim *$ to this equation, and the fact that $\tilde{\underline{a}}_{b2}^* = \underline{a}_{f2}$ and get

$$\underline{a}_{f3} = \begin{bmatrix} \underline{a}_{f2} \\ \hline 0 \end{bmatrix} - \gamma_2 \begin{bmatrix} 0 \\ \hline \tilde{\underline{a}}_{f2}^* \end{bmatrix}$$

the forward filter recursion

- Now that we have motivation, we'll follow formal derivation in Haykin. Use $\underline{a}_m = \underline{a}_{fm}$.

$$\underline{a}_m = \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} + k_m \begin{bmatrix} 0 \\ \tilde{\underline{a}}^*_{m-1} \end{bmatrix} \qquad k_m = -\gamma_{m-1}$$

<u>Step m</u>: premult by $R_{m+1} = E\left[\underline{u}_{m+1}(n)\,\underline{u}_{m+1}(n)^\dagger\right]$

$$R_{m+1}\underline{a}_m = \left[\begin{array}{c|c} R_m & \tilde{\underline{r}}^* \\ \hline \tilde{\underline{r}}^T & r(0) \end{array}\right]\begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} + k_m \left[\begin{array}{c|c} r(0) & \underline{r}^\dagger \\ \hline \underline{r} & R_m \end{array}\right]\begin{bmatrix} 0 \\ \tilde{\underline{a}}^*_{m-1} \end{bmatrix} = \begin{bmatrix} P_m \\ \hline \underline{0}_m \end{bmatrix}$$

$$\begin{bmatrix} R_m \underline{a}_{m-1} \\ \hline \tilde{\underline{r}}^T \underline{a}_{m-1} \end{bmatrix} + k_m \begin{bmatrix} \underline{r}^\dagger \tilde{\underline{a}}^*_{m-1} \\ \hline R_m \tilde{\underline{a}}^*_{m-1} \end{bmatrix} = \begin{bmatrix} P_m \\ \hline \underline{0}_m \end{bmatrix}$$

$$\begin{bmatrix} P_{m-1} \\ \hline \underline{0}_{m-1} \\ \hline \Delta_{m-1} \end{bmatrix} + k_m \begin{bmatrix} \Delta^*_{m-1} \\ \hline \underline{0}_{m-1} \\ \hline P_{m-1} \end{bmatrix} = \begin{bmatrix} P_m \\ \hline \underline{0}_m \end{bmatrix}$$

hence $\quad \Delta_{m-1} + k_m P_{m-1} = 0 \quad \Rightarrow \quad \boxed{k_m = -\dfrac{\Delta_{m-1}}{P_{m-1}}}$

$$\begin{aligned} P_m &= P_{m-1} + k_m \Delta^*_{m-1} \\ &= P_{m-1} - \frac{|\Delta_{m-1}|^2}{P_{m-1}} \\ &= P_{m-1} - |k_m|^2 P_{m-1} \end{aligned} \qquad \boxed{P_m = \left(1 - |k_m|^2\right) P_{m-1}}$$

— that's all there is to it.

$$\begin{tikzcd} P_{m-1} \arrow[rrr] \arrow[drr, bend left] & & & P_m \\ \underline{a}_{m-1} \arrow[r] & \Delta_{m-1} \arrow[r, bend left] & k_m \arrow[r, bend left] \arrow[ur, bend left] & \underline{a}_m \end{tikzcd}$$

- Some observations:
  - → $K_m$, the reflection coeff (related to PARCOR coeff) decreases to 0 as $m \to \infty$ (consider est'n of $f_m(n)$ from $b_m(n-1)$)
  - → zero order has

$$\underline{a}_0 = [1] \quad P_0 = r(0) \quad \text{(trivial case)}$$

hence $P_M = P_0 \prod_{m=1}^{M} (1 - |K_m|^2)$

  - → from the recursion

$$\underline{a}_m = \begin{bmatrix} \underline{a}_{m-1} \\ \overline{0} \end{bmatrix} + K_m \begin{bmatrix} \underline{0} \\ \underline{\tilde{a}}^*_{m-1} \end{bmatrix} = \begin{bmatrix} 1 \\ a_{m,1} \\ \\ a_{m,m-1} \\ K_m \end{bmatrix} = \begin{bmatrix} a_{m,0} \\ a_{m,1} \\ \\ a_{m,m-1} \\ a_{m,m} \end{bmatrix}$$

note last component is $K_m$

  - → from the PARCOR coefficient

$$\gamma_m = \frac{E[f_m^*(n)\, b_m(n-1)]}{E[|b_m(n-1)|^2]} = -K_{m+1}$$

$$E[f_m^*(n)\, b_m(n-1)] = -K_{m+1} P_m = \Delta_m$$

hence the intermediate quantities

$$\Delta_{m-1} = E[f_{m-1}^*(n)\, b_{m-1}(n-1)] \quad \text{the covariance}$$

$$K_m = -\frac{E[f_{m-1}^*(n)\, b_{m-1}(n-1)]}{E[|b_{m-1}(n-1)|^2]}$$

  - → $O(m)$ operations in step $m$, so $O(M^2/2)$ operations cumulative

→ Levinson-Durbin and conventional solution of Yule Walker give same result $\underline{a}_M, P_M$ but L-D generates $\underline{a}_m, P_m$ for all intermediate orders, too.

→ We went

$$\{r(0), r(1), \cdots r(M)\} \longrightarrow \begin{array}{l}\{\underline{a}_0, \underline{a}_1, \cdots \underline{a}_M\} \\ \{K_1, \cdots K_M\} \quad \{P_0, P_1, \cdots P_M\}\end{array}$$

but we can also do this;

$$\{r(0), K_1 \cdots, K_M\} \longrightarrow \begin{array}{l}\{\underline{a}_0, \underline{a}_1, \cdots \underline{a}_M\} \\ \{P_0, \cdots P_M\}\end{array}$$

since

$$\underline{a}_m = \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} + K_m \begin{bmatrix} 0 \\ \underline{\tilde{a}}^*_{m-1} \end{bmatrix}, \qquad P_m = (1 - |K_m|^2) P_{m-1}$$

- Inverse L-D

$$\underline{a}_M \longrightarrow \{\underline{a}_0, \cdots \underline{a}_M\}, \{K_1, \cdots K_M\}$$

From the highest order PEF, we can obtain those of lower order, as well as $\{K_1, \cdots K_M\}$ (which we obtain as last components of PEFs)

$$\underline{a}_m = \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} + K_m \begin{bmatrix} 0 \\ \underline{\tilde{a}}_{m-1} \end{bmatrix} \xrightarrow{\text{row } k,\ k=1 \cdots m-1} a_{m,k} = a_{m-1,k} + K_m a^*_{m-1,m-k}$$

$$\underline{\tilde{a}}^*_m = \begin{bmatrix} 0 \\ \underline{\tilde{a}}^*_{m-1} \end{bmatrix} + K^*_m \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} \xrightarrow{\text{row } m-k} a^*_{m,m-k} = a^*_{m-1,m-k} + K^*_m a_{m-1,k}$$

$$\begin{bmatrix} a_{m,k} \\ a^*_{m,m-k} \end{bmatrix} = \begin{bmatrix} 1 & K_m \\ K^*_m & 1 \end{bmatrix} \begin{bmatrix} a_{m-1,k} \\ a^*_{m-1,m-k} \end{bmatrix}$$

solve for lower order coefficients from higher

invert

$$\frac{1}{1-|K_m|^2} \begin{bmatrix} 1 & -K_m \\ -K^*_m & 1 \end{bmatrix}$$

$$a_{m-1,k} = \frac{a_{m,k} - K_m a^*_{m,m-k}}{1 - |K_m|^2} = \frac{a_{m,k} - a_{m,m} a^*_{m,m-k}}{1 - |K_m|^2}$$

- Recovery of autocorrelation function: in $\boxed{\text{Haykin 6.4}}$

From $\{P_0, K_1 \cdots K_m\}$ we can recover the $\{r(0), r(1), \cdots, r(M)\}$.

Just consider the Yule Walker equations of order $m$

$$\begin{bmatrix} r(0) & r(1) & \cdots & r(m-1) & r(m) \\ r^*(1) & r(0) & & & r(m-1) \\ & & \ddots & & \\ r^*(m) & r^*(m-1) & \cdots & r^*(1) & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_{m,1} \\ \vdots \\ a_{m,m-1} \\ K_m \end{bmatrix} = \begin{bmatrix} P_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Solution 1: Use row 1 to express $r(m)$ in terms of lower:

$$r(0) + a_{m,1} r(1) + \cdots + a_{m,m-1} r(m-1) + K_m r(m) = P_m = (1-|K_m|^2) P_{m-1}$$

$$r(m) = \frac{(1-|K_m|^2) P_{m-1}}{K_m} - \frac{1}{K_m} \sum_{k=0}^{m-1} a_{m,k} r(k)$$

So given $P_0$ $(= r(0))$, $\underline{a}_0$ $(= [1])$, we form

$\underline{a}_1$, $r(1)$, $P_1$; $\underline{a}_2$, $r(2)$, $P_2$; etc.

Solution 2: Use row $m$ to express $r(m)$ in terms of lower

$$r^*(m) = -a_{m,1} r^*(m-1) - \cdots - a_{m,m-1} r^*(1) - a_{m,m} r(0)$$

$$r(m) = -\sum_{k=1}^{m} a^*_{m,k} r(m-k) \qquad \left( = -\sum_{k=1}^{m-1} a^*_{m,k} r(m-k) - K^*_m P_0 \right)$$

So given $P_0$, $\underline{a}_0$, do this:

$\underline{a}_1$, $r(1)$; $\underline{a}_2$, $r(2)$; $\cdots$

Both require the $\{\underline{a}_m\}$, which are generated from the $\{K_m\}$ through the basic recursion.

- Summary of relations:
  A process is characterized by the shape and size of its power spectrum ( $\underline{a}_M$ and $r(0)$, respectively ). Only $M+1$ parameters for AR, order $M$.

$$\begin{tikzcd} r(0), \dots, r(M) \arrow[r, bend left] & \begin{array}{c} \underline{a}_0, \underline{a}_1, \dots, \underline{a}_M \\ K_1, \dots, K_M \\ P_0, P_1, \dots P_M \end{array} \arrow[l, bend left] \end{tikzcd}$$

$$\begin{tikzcd} K_1, \dots, K_M \arrow[r, bend left] & \underline{a}_0, \underline{a}_1, \dots \underline{a}_M \arrow[l, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \underline{a}_M \arrow[r, bend left] & \underline{a}_0, \underline{a}_1, \dots, \underline{a}_M \end{tikzcd}$$

$$\begin{tikzcd} P_0, K_1, \dots, K_M \arrow[r, bend left] & P_0, P_1, \dots P_M \arrow[l, bend left, "?"] \end{tikzcd}$$

→ Can you think of another common set of parameters?