8.5 Recursion for the Sum of Squared Errors

- We have defined
$$E(n) = \sum_{i=1}^{n} \lambda^{n-i} |d(i) - \hat{d}(i)|^2$$
We can similarly define
$$E_d(n) = \sum_{i=1}^{n} \lambda^{n-i} |d(i)|^2 \quad , \quad E_{\hat{d}}(n) = \underline{w}^+(n)\, \Phi(n)\, \underline{w}(n)$$
and when $\underline{w}(n) = \hat{\underline{w}}(n) = \Phi^{-1}(n)\, \underline{z}(n)$, Pythagoras applies
$$E_{min}(n) = E_d(n) - \hat{\underline{w}}^+(n)\,\Phi(n)\,\hat{\underline{w}}(n) = E_d(n) - \underline{z}^+(n)\,\Phi^{-1}(n)\,\underline{z}(n)$$
$$= E_d(n) - \hat{\underline{w}}^+(n)\,\underline{z}(n) = E_d(n) - \underline{z}^+(n)\,\hat{\underline{w}}(n)$$

- We have recursions for everything else. Now find a recursion for $E_{min}(n)$ in terms of $E_{min}(n-1)$. First, some reminders:
$$E_d(n) = \lambda E_d(n-1) + |d(n)|^2$$
$$\underline{z}(n) = \lambda \underline{z}(n-1) + d^*(n)\,\underline{u}(n)$$
$$\hat{\underline{w}}(n) = \hat{\underline{w}}(n-1) + \underline{k}(n)\,\xi^*(n)$$
$$\xi(n) = d(n) - \underline{w}^+(n-1)\,\underline{u}(n)$$
$$e(n) = d(n) - \underline{w}^+(n)\,\underline{u}(n)$$

- Start with

$$E_{min}(n) = E_d(n) - \underline{z}^+(n)\,\hat{\underline{w}}(n)$$

$$= \lambda E_d(n-1) + |d(n)|^2 - \left(\lambda \underline{z}^+(n-1) + d(n)\,\underline{u}^+(n)\right)\left(\hat{\underline{w}}(n-1) + \underline{k}(n)\,\xi^*(n)\right)$$

$$= \lambda E_d(n-1) + |d(n)|^2 - \left(\lambda \underline{z}^+(n-1) + d(n)\,\underline{u}^+(n)\right)\hat{\underline{w}}(n-1) - \underline{z}(n)\,\underline{k}(n)\,\xi^*(n)$$

$$= \lambda\left(E_d(n-1) - \underline{z}^+(n-1)\,\hat{\underline{w}}(n-1)\right) + d(n)\left(d^*(n) - \underline{u}^+(n)\,\hat{\underline{w}}(n-1)\right) - \underline{z}(n)\,\underline{k}(n)\,\xi^*(n)$$

$$= \lambda E_{min}(n-1) + d(n)\,\xi^*(n) - \underline{z}(n)\,\Phi^{-1}(n)\,\underline{u}(n)\,\xi^*(n)$$

$$= \lambda E_{min}(n-1) + \left(d(n) - \hat{\underline{w}}^+(n)\,\underline{u}(n)\right)\xi^*(n)$$

$$= \lambda E_{min}(n-1) + e(n)\,\xi^*(n)$$

They are all equivalent, but the last one is very interesting.

- Are the a priori, a posteriori errors related?

$$e(n) = d(n) - \hat{\underline{w}}^+(n)\,\underline{u}(n)$$

$$= d(n) - \left(\hat{\underline{w}}(n-1) + \underline{k}(n)\,\xi^*(n)\right)^+\underline{u}(n)$$

$$= \underbrace{d(n) - \hat{\underline{w}}^+(n-1)\,\underline{u}(n)}_{\xi(n)} - \underline{k}^+(n)\,\underline{u}(n)\,\xi(n)$$

$$= \left(1 - \underline{k}^+(n)\,\underline{u}(n)\right)\xi(n)$$