PRACTICAL GUIDE TO ERRORS AND ERROR PROPAGATION

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A. Definition: The error on a given number is an estimate of the range within which the actual value of that number should fall.

A given measurement and its error are represented by

$$Y \pm \Delta y$$

where Y = the number giving a measured or calculated value of some parameter e.g. length of an object, current through a wire etc.

Δy = the estimated error on y's value of the parameter

<u>Example</u>

Consider the measurement of the thickness (call it t) of a sheet of metal using a ruler $t_{exp\ 1} = 6.5 \pm 0.5$ mm.

This means the true value of t probably lies within the range 6.0 mm ($t_{exp\ 1} - \Delta t$) to 7.0 mm ($t_{exp\ 1} + \Delta t$). Statistically speaking 68% of the time the actual t value lies somewhere within that range, 95% of the time within twice that range ($t_{exp} \pm 2\Delta t$) and 99.7% of the time within triple the range ($t_{exp} \pm 3\Delta t$).

If one makes a more accurate measurement using a vernier caipers on this sheet the result is $t_{exp\ II} = 6.43 \pm .02$mm. Thus, as expected, the more accurate estimate, $t_{exp\ II}$, of t's true value lies within the error range of $t_{exp\ I}$.

Similarly using a micrometer yields $t_{exp\ 111}$ 6.416 ± .001 mm, within the error range of both $t_{exp\ 1}$ and $t_{exp\ 11}$.

Note that an error is usually only expressed to one or two significant figures. Also the number is only given to the same level of accuracy as the error i.e. if error is stated in tenths of a mm (as in the first example) then the smallest unit used in the number should be tenth of a mm. There are two ways of giving the error on a number:

- the absolute error where error itself has the same units as the number. This is the type of error shown in the examples given so far e.g. 6.5 ± 0.5 mm. It is symbolized by $y \pm \Delta y$;
- the relative error where the error is given as a fraction or percentage of the actual number. Thus the error is unitless. These are calculated as follows:

$$\text{fractional error} \pm \frac{\Delta y}{y}$$

$$\text{percentage error} \pm \frac{\Delta y}{y} \times 100\%$$

where y is number and Δy its absolute error. e.g. The absolute error 6.5 ± 0.5 mm can also be given as 6.5 mm ± 8%. (Fractional errors are not usually used in an error statement).

B. <u>Propagation of Errors</u>

Whenever measurements with errors are used within a calculation the final result of that calculation will itself have an error range. In other words the errors on a number must be propagated through any formula it is used within. The three rules for error propagation follow.

(1) When adding or subtracting numbers add the <u>absolute</u> errors and thus produce the <u>absolute</u> error of the result

e.g. if $z = y + x$ where $y = y_{exp} \pm \Delta y$ $\quad x = x_{exp} \pm \Delta x$

$$\text{then } Z = (y_{exp} + x_{exp}) \pm \Delta Z$$

$$\Delta Z = \Delta x + \Delta y$$

Similarly if $A = y - x$

$$\text{then } A = (y_{exp} - x_{exp}) \pm \Delta A$$

$$\Delta A = \Delta y + \Delta x$$

This is called worst case error analysis. That is, one takes the largest possible error which would occur if the true y and x value lie at the extremes of the error ranges, e.g. let y = 6.5 ± 0.5 mm and x = 2.0 ± 0.1 mm then for one extreme of A let y actually = $y_{exp} - \Delta y$ = 6.0 mm and x actually = $x_{exp} + \Delta x$ = 2.1 mm Then the lower extreme value of A is

$$A_{lower} = (y_{exp} - \Delta y) - (x_{exp} + \Delta x)$$

$$= 6.0 - 2.1 = 3.9 \text{ mm}$$

Similarly for the upper extreme

$$A_{upper} = (y_{exp} + \Delta y) - (x_{exp} - \Delta x)$$

$$= (6.5 + 0.5) - (2.0 - 0.1) = 5.1 \text{ mm}$$

A itself is = $Y_{exp} - x_{exp} = 6.5 - 2.0 = 4.5$ mm

Then $+ \Delta A = A_{upper} - A = 5.1 - 4.5 = +.06$ mm

$- \Delta A = A_{lower} - A = 3.9 - 4.5 = - 0.6$ mm

Thus A - 4.5 ± 0.6 (this yields the proper range of 3.9 to 5.1).

In an actual calculation all that is shown is

$A = Y_{exp} - x_{exp} = 6.5 - 2.0 = 4.5$ mm

$\Delta A = \Delta y + \Delta x = 0.5 + 0.1 = 0.6$ mm

$A = 4.5 \pm 0.6$ mm

Similarly $Z = y_{exp} + x_{exp} = 6.5 + 2.0 = 8.5$ mm

$\Delta Z = \Delta y + \Delta x = 0.5 + 0.1 = 0.6$ mm

$Z = 8.5 \pm 0.6$ mm

e.g. A sheet of metal is coated on one side with a film 0.135 ± .005 mm thick. If the total thickness of the film plus metal is 7.01 ± .01 mm thick, what is the thickness of the metal itself?

Answer:

let F = the film thickness

M = the metal thickness

T = the total thickness

Then $M = T - F = 7.010 - 0.135 = 6.875$ mm

$\Delta M = \Delta T + \Delta F = 0.01 + 0.005 = 0.015$ mm

Thus $M = 6.88 \pm 0.02$ mm

Note the round off in the final result to bring all portions to the same accuracy. However, if M is used in other calculations use the unrounded values in thos combinations and round the final results.

(II) When multiplying or dividing add the relative errors and thus produce the relative error of the result

e.g. let $B = y.x$

$y = y_{exp} \pm \Delta y$

$x = x_{exp} \pm \Delta x$

then $B = y_{exp} \cdot x_{exp} \pm \Delta B$

The fractional error $\frac{\Delta B}{B} = \frac{\Delta y}{y_{exp}} + \frac{\Delta x}{x_{exp}}$

One could obtain a similar result if percentage errors are used i.e.

$$\frac{\Delta B}{B} \text{ x } 100\% = \frac{\Delta y}{y_{exp}} \text{ x } 100\% + \frac{\Delta x}{x_{exp}} \text{ x } 100\%$$

e.g. let $D = \frac{y}{x}$

then $D = \frac{y_{exp}}{x_{exp}} \pm \Delta D$

The fraction error is $\frac{\Delta D}{D} = \frac{\Delta y}{y_{exp}} + \frac{\Delta x}{x_{exp}}$

A similar result is obtained with percentage errors.

This rule produces the worst case as in part I

e.g. Find the volume of a slab where

width W = 5.4 ± 0.5 cm

height H = 0.23 ± 0.01 mm

length L = 15.3 ± 0.5 cm

Answer:

Restating H = 0.23 ± 0.01 mm

= 0.023 ± 0.001 cm

then the volume V = W.H.L.

= (5.4)(0.023)(15.3)

= 1.900 cm^3

$$\frac{\Delta V}{V} = \frac{\Delta W}{W} + \frac{\Delta H}{H} + \frac{\Delta L}{L}$$

$$= \frac{0.5}{5.4} + \frac{0.01}{0.23} + \frac{0.5}{15.3}$$

$= 0.093 + 0.043 + 0.033$

$= 0.169$

or $\frac{\Delta V}{V}$ x 100% = 16.9%

V = 1.90 cm ± 17%

= 1.90 ± 0.31 cm

= 1.9 ± 0.3 cm

(note the round off of the result again).

e.g. A car covers a distance D = 28.4 ± 0.01 m in a time t = 1.1 ± 0.1 sec. Find the car's speed.

Answer: the speed S = $\frac{d}{t}$

$$= \frac{28.4}{1.1}$$

= 25.82 m/sec

$$\frac{\Delta S}{S} = \frac{\Delta D}{D} + \frac{\Delta t}{t}$$

$= 0.00035 + 0.091$

$= 0.091$

the relative error is $\frac{\Delta S}{S}$ x 100% = 9.1%

Thus the speed is S = 25.8 m ± 9%

= 25.8 ± 2.4 m/sec

= 26 ± 2 m/sec

Note that the error in this case is totally dominated by the error on the time. Thus one could improve the experiment simply by improving the timing accuracy.

(III) Errors of a general function:

Let $G = f(x)$ with $x = x_{exp} \pm \Delta x$

where $f(x)$ is any arbitrary function

then $\Delta G = \left.\frac{df(x)}{dx}\right|_{x=x_{exp}} \Delta x$

Thus the error on G is approximately equal to the derivative of f(x) with respect to x, (evaluated at $x = x_{exp}$) times the error on x.

e.g. let $j = x^{10}$ $x = 5.0 \pm 0.2$ m

$$= x_{exp}^{10}$$

$$= 9.77 \times 10^6 \text{ m}^{10}$$

$$\Delta J = \left.\frac{dx^{10}}{dx}\right|_{x=x_{exp}} \Delta x$$

$$= (10x^9_{exp})\Delta x$$
$$= 10(5^9)\ 0.2$$
$$= 2(1.95 \times 10^6)$$
$$= 3.9 \times 10^6 \text{ m}^{10}$$

Note: this is the same result as could be obtained by using rule II. J is produced by multiplying x by itself 10 times.

Thus if $x = 5.0 \pm 0.2$ m $= 5.0$ m $\pm 4\%$

$$J = 9.8 \times 10^6 \pm (10 \times 4\%)$$
$$= 9.8 \times 10^6 \pm 40\%$$
$$= (9.8 \pm 3.9) \times 10^6$$

This derivative rule can be used to produce either rule I or rule II. However it is usually easier to apply those rules than to employ this general purpose one for simple equations.

This general rule is actually an approximation to the true error given by

$$G = f(x)$$
$$+\Delta G = f(x_{exp} + \Delta x) - f(x_{exp})$$

$$-\Delta G = f(x_{exp} - \Delta x) - f(x_{exp}).$$

The derivative procedure will yield about the same value of ΔG as this exact formula if the relative error on x is small ($\Delta x < 15\%$). It is also a much easier rule to use when a number of x_{exp} are to be evaluated.

For the trigometric functions there is one small trick that must be used whem employing the derivative method. Due to the nature of these functions the error, Δx, must be converted to radians, no matter what units x_{exp} is expressed in

e.g. Let $K = \cos x \qquad x = 30^\circ \pm 1^\circ$

$$= \cos 30^\circ$$

$$= 0.866$$

$$\Delta K = \frac{d \cos x}{dx}\Big|_{x=30^\circ} \Delta x$$

$$= (-\sin x)\, \Delta x$$

Converting Δx to radians $\Delta x = \dfrac{(\Delta x)\pi}{180}$

$$= 0.0175$$

Also, since the error, Δx, has both positive and negative values thus the negative sign of the derivative may be eliminated.

Since $\Delta K = (\sin 30^\circ)\ (0.0175)$

$$= (0.500)\ (0.0.175)$$

$$= 0.0087$$

then $K = 0.866 \pm 0.009$.

Note that the radian conversion of Δx produces a dimensionless quantity for ΔK, to match the dimensionless value cos (x) yields for K (radians have not units whereas an angle in degrees has the unit of degrees).

Finally when the function involves several numbers with error, this rule can be extended as follows:

e.g. If Z - f(x,y) $X = x_{exp} \pm \Delta x$ $y = y_{exp} \pm \Delta y$

$$\text{then } \Delta Z = \frac{df}{dx} \Delta x + \frac{df}{dy} \Delta y.$$

Here ΔZ is obtained from the total derivative of f(x,y).

e.g. Consider again the volume problem given in rule II. Then using the derivative rule

$$V = W.H.\,L$$

$$= 1.900 \text{ cm}^3 \text{ as before}$$

$$\Delta V = \frac{dV}{dV} \Delta W + \frac{dV}{dH} \Delta H + \frac{dV}{dL} \Delta L$$

$$= HL\Delta W + WL\Delta H + WH\Delta L$$

Since $W = 5.4 \pm 0.5$ cm

$H = 0.023 \pm 0.001$ cm

$L = 15.3 \pm 0.5$ cm

then $\Delta V = (0.023)(15.3)(0.5)$

$+ (5.4)(15.3)(0.001)$

$+ (5.3)(0.023)(0.5)$

$= 0.176 + 0.083 + 0.062$

$= 0.321 \text{ cm}^3$

Thus $V = 1.9 \pm 0.3 \text{ cm}^3$ as was obtained before.

C. <u>Some Helpful Hints</u>

(a) When trying to determine if two numbers are the same, subtract their values and compare the resulting difference to the expected error range.

e.g. If the measured period, P, of a pendulum is $6.01 \pm .01$ sec and the theoretically calculated value is 5.99 ± 0.02 sec, does the theory agree with the given data?

The difference $D = P_{theory} - P_{exp}$

$= 5.99 - 6.01$

$= 0.02$ sec.

$$\Delta D = \Delta P_{theory} + \Delta P_{exp}$$
$$= 0.02 + 0.01$$
$$= 0.03 \text{ sec}$$
$$D = -\,0.02 \pm 0.03 \text{ sec.}$$

Since this difference is less than the error then the theoretical and experimental periods are said to agree within the experimental error.

(b) When multiplying or dividing any measurement by a pure number (i.e. one without an error) the error is obtained by performing the same operation upon the measurement's error.

e.g. 20 oscillations of a pendulum occur in 40.7 ± 0.2 sec. What is the period of the pendulum?

Let the time 5 = 40.7 ± 0.2 sec

the number of cycles C = 20

$$\text{Then the period } P = \frac{t}{c}$$

$$= \frac{40.7}{20} \pm \frac{0.2}{20}$$

$$= 2.04 \pm 0.01 \text{ sec.}$$

Using Rule II, since C has no error, then the relative error must be the same for both the time t and the period P. Thus to maintain this relative error identical operations (in this case a division by C) must be performed on both t_{exp} and Δt. Note that this may be used as a technique to reduce the error on the period. In the above problem the absolute error on the time value is often independent of the length of time measured. Thus if only one oscillation was taken with t = 2.1 ± .2 sec = the period, then the absolute error is 20 times that obtained in the above calculation, where many periods are measured.