Solving the Second Order Systems

- Continuing with the simple RLC circuit

(4) Make the assumption that solutions are of the exponential form:

$$i(t) = A\,exp(st)$$

- where A and s are constants of integration.
- Then substituting into the differential equation

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

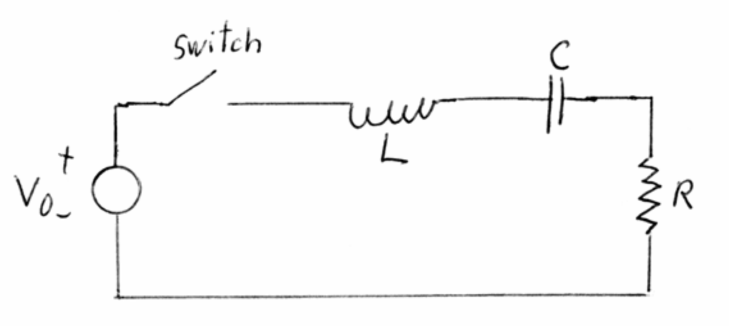
$$L\frac{d^2 A\,exp(st)}{dt^2} + R\frac{dA\,exp(st)}{dt} + \frac{1}{C}A\,exp(st) = 0$$

$$Ls^2 A\,exp(st) + RsA\,exp(st) + \frac{A}{C}exp(st) = 0$$

- Dividing out the exponential for the characteristic equation

$$Ls^2 + Rs + \frac{1}{C} = 0$$

- Also called the Homogeneous equation
- Thus quadratic equation and has generally two solutions.
- There are 3 types of solutions
- Each type produces very different circuit behaviour
- Note that some solutions involve complex numbers.



General solution of the Second Order Systems

- Consider the characteristic equations as a quadratic
- Recall that for a quadratic equation:

$$ax^2 + bx + c = 0$$

- The solution has two roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Thus for the characteristic equation

$$Ls^2 + Rs + \frac{1}{C} = 0$$

- or rewriting this

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Thus

$$a = 1 \qquad b = \frac{R}{L} \qquad c = \frac{1}{LC}$$

- The general solution is:

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

General solution Second Order Systems Cont'd

- Second order equations have two solutions
- Usually define

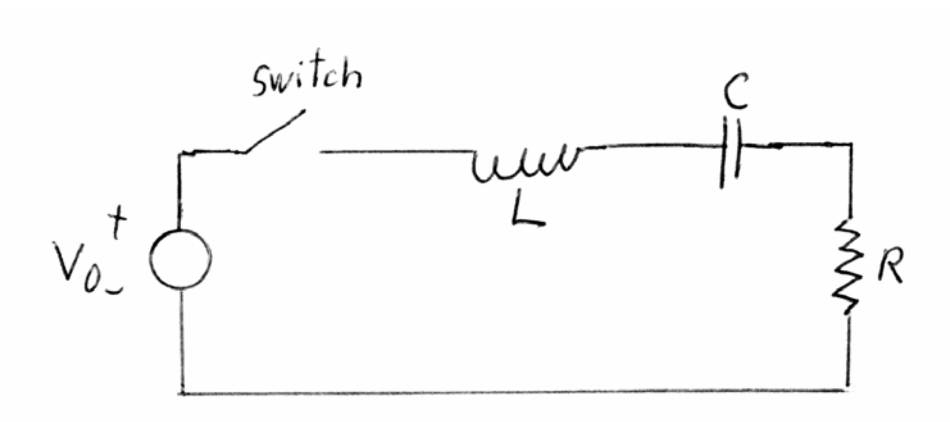
$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

- The type of solutions depends on the value these solutions
- The type of solution is set by the **Descriminant**

$$D = \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right]$$

- Recall $\frac{R}{L}$ is the time constant of the resistor inductor circuit
- Clearly the descriminate can be either positive, zero, or negative



3 solutions of the Second Order Systems

- What the **Descriminant** represents is about energy flows

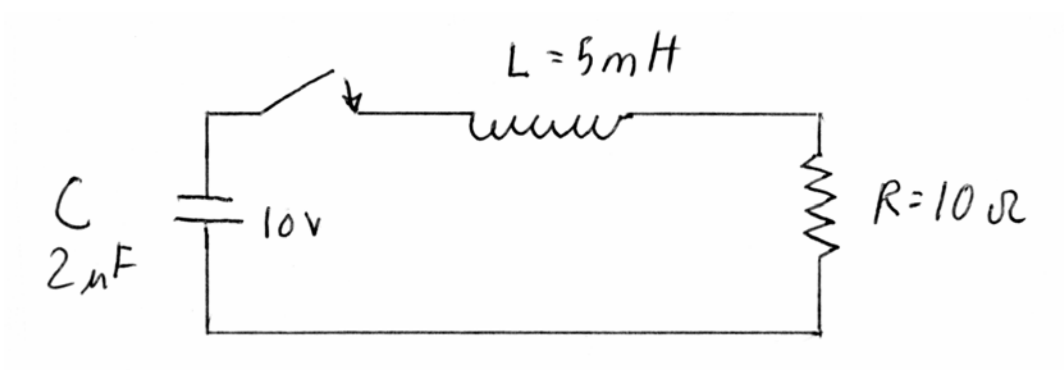
$$D = \left[\left(\frac{R}{2L} \right)^2 - \frac{1}{LC} \right]$$

- How fast is energy transferred from the L to the C
- How fast is energy lost to the resistor
- There are three cases set by the descriminant

- D > 0 : roots real and unequal
- In electronics called the overdamped case

- D = 0 : roots real and equal
- In electronics the critically damped case

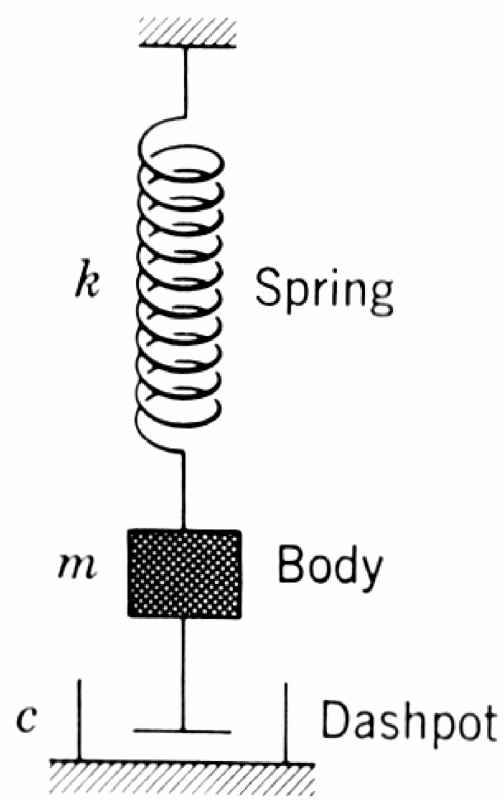
- D < 0 : roots complex and unequal
- In electronics: the underdamped case: very important



Second Order Solutions

- Second order equations are all about the energy flow
- Consider the spring case
- The spring and the mass have energy storage
- The damping pot losses the energy
- The critical factor is how fast is energy lost

- In **Overdamped** the energy is lost very fast
- The block just moves to the rest point
- **Critically Damped** the loss rate is smaller
- Just enough for one movement up and down
- For **Underdamped** spring moves up and down
- Energy is transferred from the mass to the spring and back again
- Loss rate is smaller than the time for transfer



Complex Numbers

- Imaginary numbers necessary for second order solutions
- Imaginary number j

$$j = \sqrt{-1}$$

- Note: in math imaginary number is called i
- But i means current in electronics so we use j
- Complex numbers involve Real and Imaginary parts

$$\vec{W} = R_W + jI_W$$

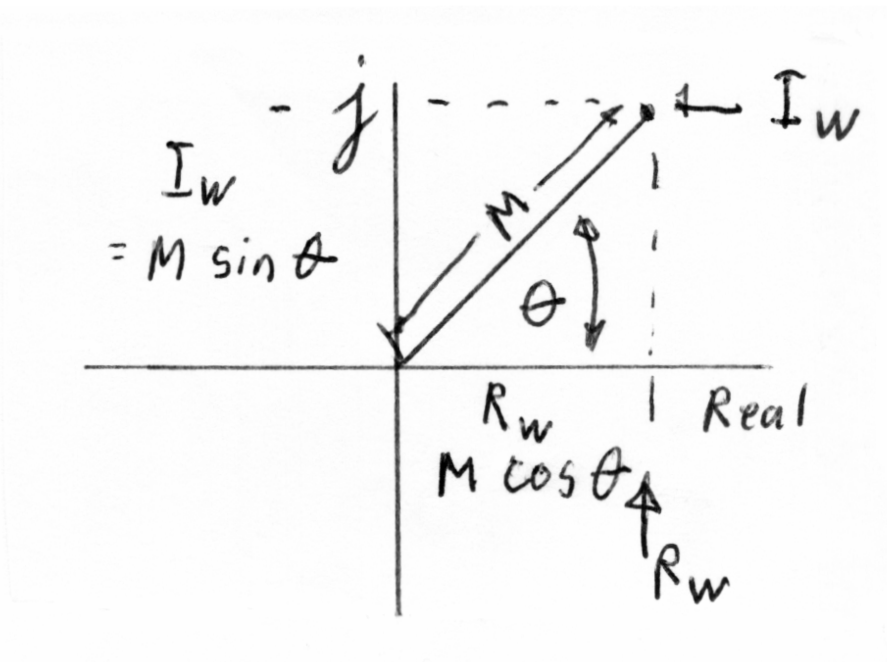
- May designate this in a vector coordinate form:

$$\vec{W} = (R_W, I_W)$$

- Example:

$$\vec{W} = 1 + j2 = (1,2)$$

$$\text{Re}(W) = 1 \qquad \text{Im}(W) = 2$$



Complex Numbers Plotted

- In electronics plot on X-Y axis
- X axis real, Y axis is imaginary
- A vector represents the imaginary number has length
- Vector has a magnitude M
- Vector is at some angle θ to (theta) the real axis
- Then the real and imaginary parts are

$$\mathrm{Re}\,al(W) = R_W = M\ \cos(\theta)$$

$$\mathrm{Im}\,aginary(W) = I_W = M\ \sin(\theta)$$

$$\vec{W} = M\ [\cos(\theta) + j\ \sin(\theta)]$$

- The magnitude

$$Mag(\vec{W}) = |\vec{W}| = \sqrt{(R_w^2 + I_w^2)}$$

- The angle

$$\theta = \arctan\left(\frac{I_W}{R_W}\right)$$

Thus can give the vector in polar coordinates

$$\vec{W} = (M, \theta) = M\angle\theta$$

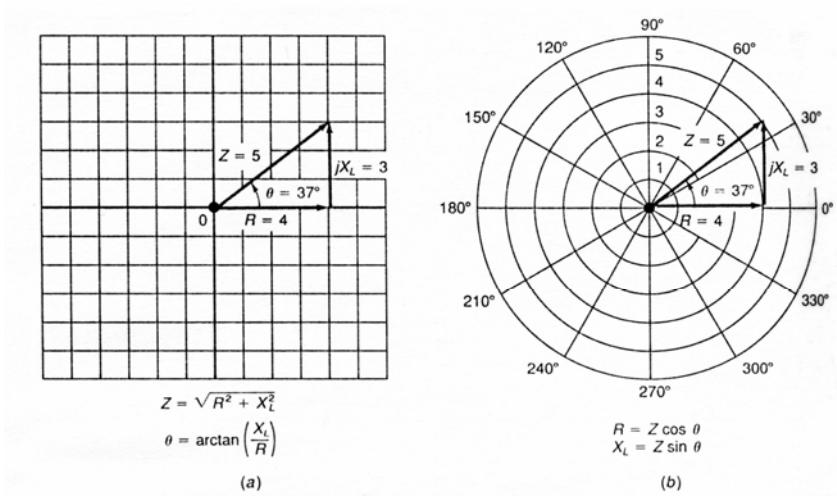


Fig. 25-8 Magnitude and angle of a complex number. (*a*) Rectangular form. (*b*) Polar form.

Complex Numbers and Exponentials

- Polar coordinates are connected to complex numbers in exp
- Consider an exponential of a complex number

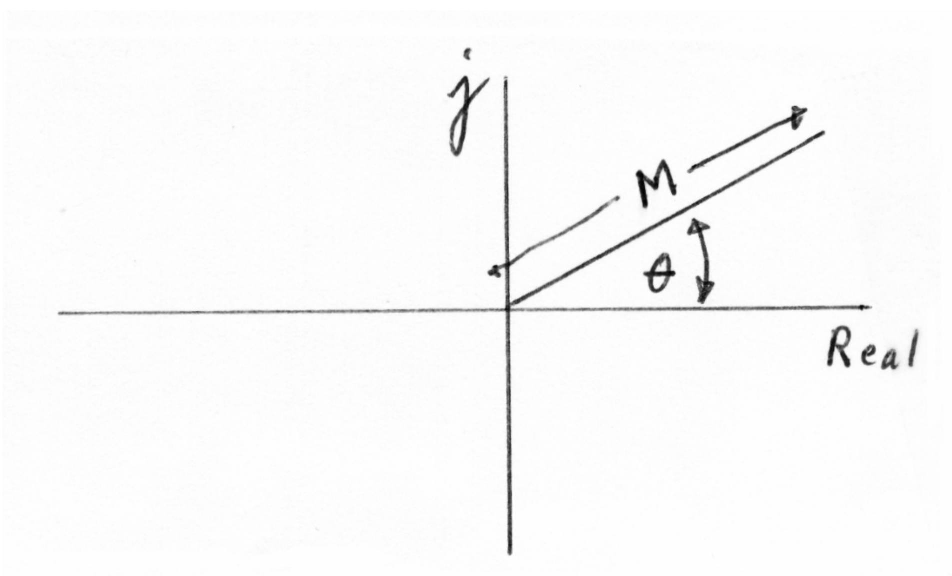
$$\vec{W} = (M, \theta) = R_w + jI_w$$

- This is given by the Euler relationship

$$\exp(j\theta) = [\cos(\theta) + j\sin(\theta)]$$

$$\vec{W} = M\exp(j\theta) = M[\cos(\theta) + j\sin(\theta)]$$

- This relationship is very important for electronics
- Used in second order circuits all the time.



Overdamped RLC

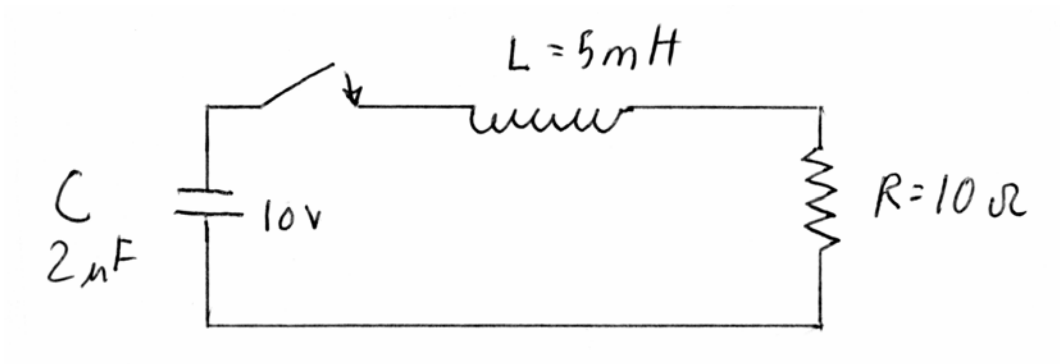
- This is a very common case
- In RLC series circuits this is the large resistor
- Energy loss in the resistor much greater than energy transfers
- Two real roots to the characteristic equation

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

- The solution is a double exponential decay

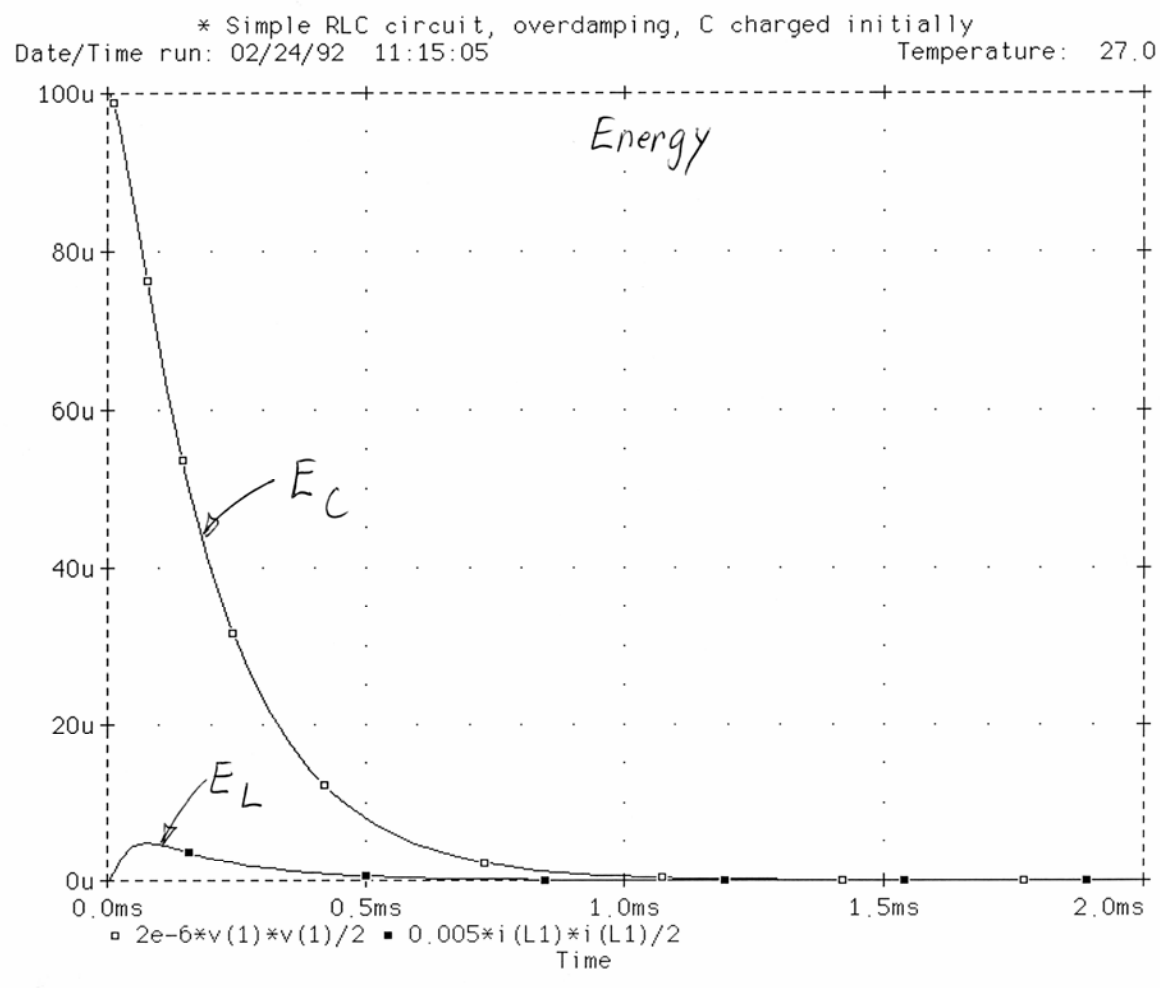
$$i(t) = A_1 \exp(s_1 t) + A_2 \exp(s_2 t)$$

- As R is increased damping increases
- Get the overdamped case



Overdamped RLC Energy Flows

- For the example case L = 5 mH, C = 2 μF, R = 200 Ω
- Also assume C is charged to 10 V at t = 0
- Initially energy starts in the Capacitor C
- Some energy transfers to the Inductor L
- C looses charge much faster than L gains current
- So energy starts to rise in L but only to a limited level
- Then energy is removed from both by the resistor



Overdamped RLC Initial Conditions

- For the overdamped case the s's are real & different

$$i(t) = A_1 \exp(s_1 t) + A_2 \exp(s_2 t)$$

- To solve the constants A need the initial conditions
- For second order need two conditions
- Thus both initial current & its derivative
- This varies from circuit to circuit
- In the case of a charged C switched into the circuit
- Since L acts as an open initially then i(0) = 0, thus

$$i(0) = A_1 + A_2$$

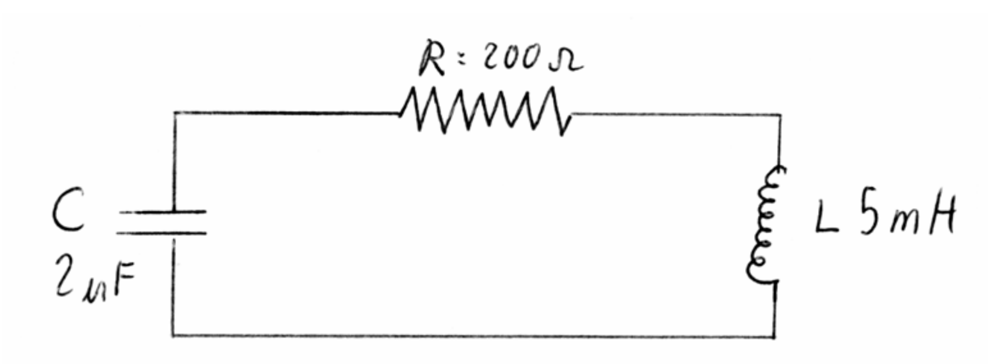
- Thus

$$A_2 = -A_1$$

- Again since the inductor acts open at time zero & i(0)=0
- Thus voltage drop across the resistor is zero

$$V_C(0) = V_L(0) = L\frac{di(0)}{dt}$$

$$\frac{di(0)}{dt} = \frac{V_c(0)}{L}$$



Overdamped RLC Full Solution

- Now using substituting A equation into the exp equation

$$i(t) = A_1 \exp(s_1 t) - A_1 \exp(s_2 t)$$

- To solve the constants A with the derivative initial condition

$$\frac{di(t)}{dt} = A_1 [s_1 \exp(s_1 t) - s_2 \exp(s_2 t)]$$

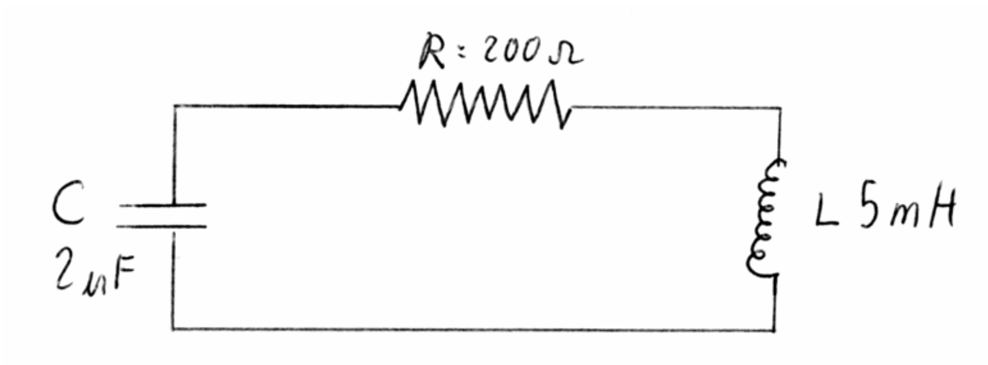
- Now applying the initial condition derivative

$$\frac{di(t=0)}{dt} = A_1 [s_1 \exp(0) - s_2 \exp(0)] = A_1 [s_1 - s_2] = \frac{V_c(0)}{L}$$

- Now solving the equations

$$A_1 = \frac{V_C(0)}{[s_1 - s_2]L}$$

$$i(t) = \frac{V_C(0)}{[s_1 - s_2]L} [\exp(s_1 t) - \exp(s_2 t)]$$



Overdamped RLC Circuit Example

- For the example case L = 5 mH, C = 2 μF, R = 200 Ω
- Solving for the roots first what are the discriminate values

$$\left(\frac{R}{2L}\right) = \frac{200}{2\times 0.005} = 2\times 10^{4}\ \sec^{-1} = \frac{1}{\tau} \qquad \tau = 50\ \mu\sec$$

$$\frac{1}{LC} = \frac{1}{0.005\times\left(2\times 10^{-6}\right)} = 10^{8}\ \sec^{-2}$$

$$D = \left[\left(\frac{R}{2L}\right)^{2} - \frac{1}{LC}\right] = \left[\left(2\times 10^{4}\right)^{2} - 10^{8}\right] = 3\times 10^{8}\ \sec^{-2}$$

- Thus gives

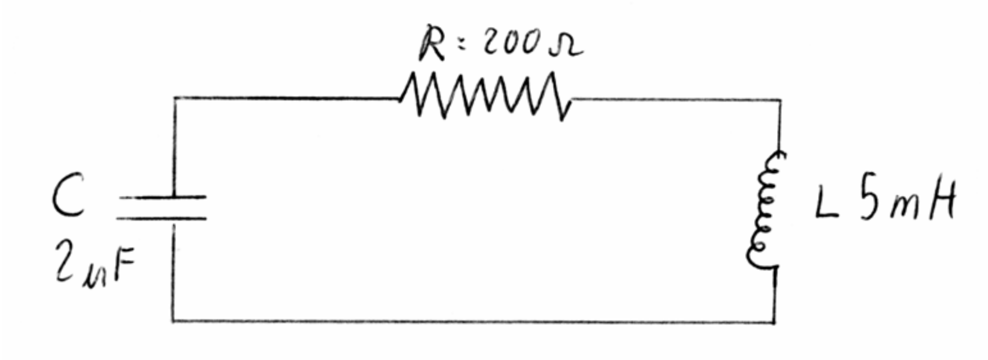
$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^{2} - \frac{1}{LC}} = -\frac{200}{2\times 0.005} + \sqrt{3\times 10^{8}} = -2.68\times 10^{3}\ \sec^{-1}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^{2} - \frac{1}{LC}} = -\frac{200}{2\times 0.005} - \sqrt{3\times 10^{8}} = -3.73\times 10^{4}\ \sec^{-1}$$

- Thus

$$A_1 = \frac{V_C(0)}{\left[s_1 - s_2\right]L} = \frac{10}{\left[1.87\times 10^{4} + 5.87\times 10^{4}\right]0.005} = 5.77\times 10^{-2}\ A$$

$$i(t) = 5.77\times 10^{-2}\left[\exp(-2.68\times 103) - \exp\left(-3.73\times 10^{4}\right)\right]A$$



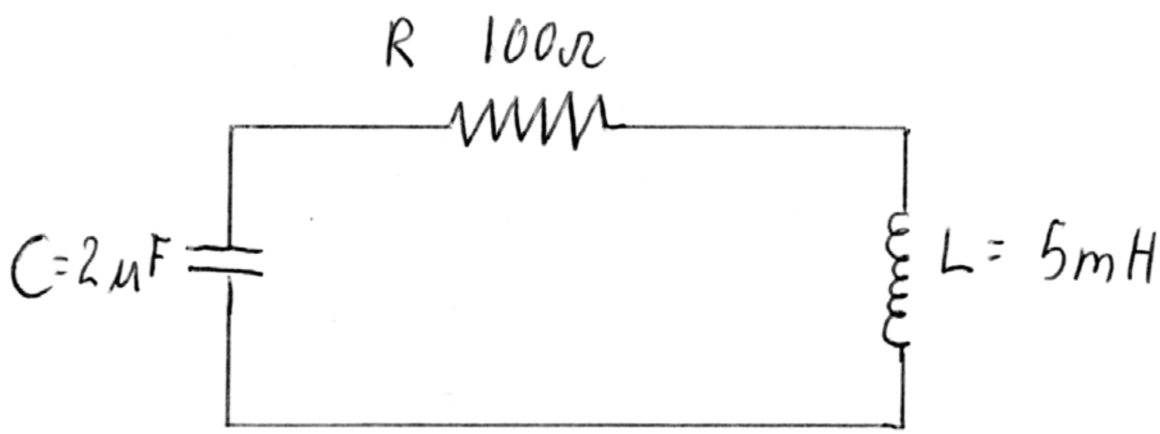
Critical Damped RLC

- If we decrease the damping (resistance) energy loss decreases
- Change the exponential decay until discriminant=0

$$D = \left[\left(\frac{R}{2L} \right)^2 - \frac{1}{LC} \right] = 0$$

$$\left(\frac{R}{2L} \right)^2 = \frac{1}{LC}$$

- This is the point called critical damping
- Difficult to achieve: only small change in R moves from this point
- Small temperature change will cause that to occur
- Energy transfer from C to L is now smaller than loss in R



Critical Damped RLC Solutions

- The characteristic equation has two identical solutions

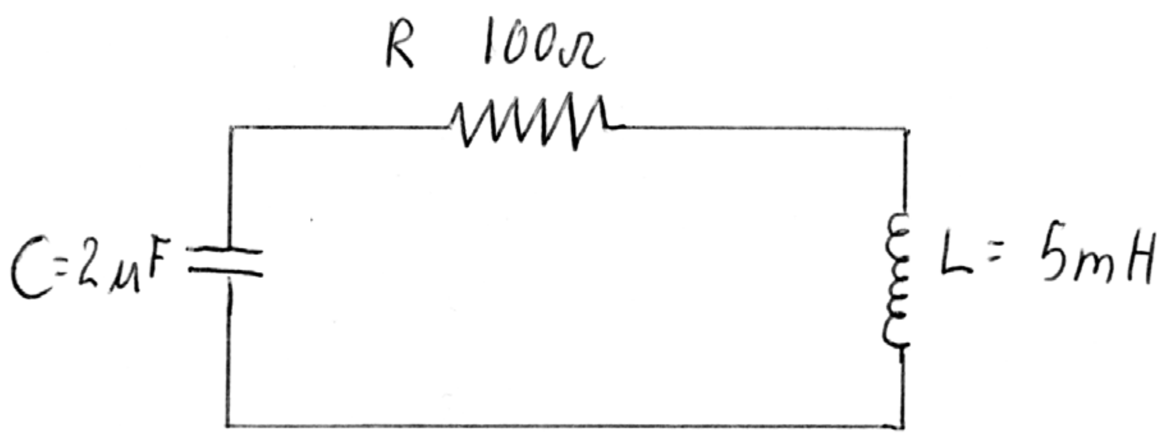
$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_1 = s_2 = -\frac{R}{2L}$$

- This is a special case, with special solution

$$i(t) = [A_1 + A_2 t]\exp\left(-\frac{Rt}{2L}\right)$$

Why this solution?
Given in the study differential equations in math

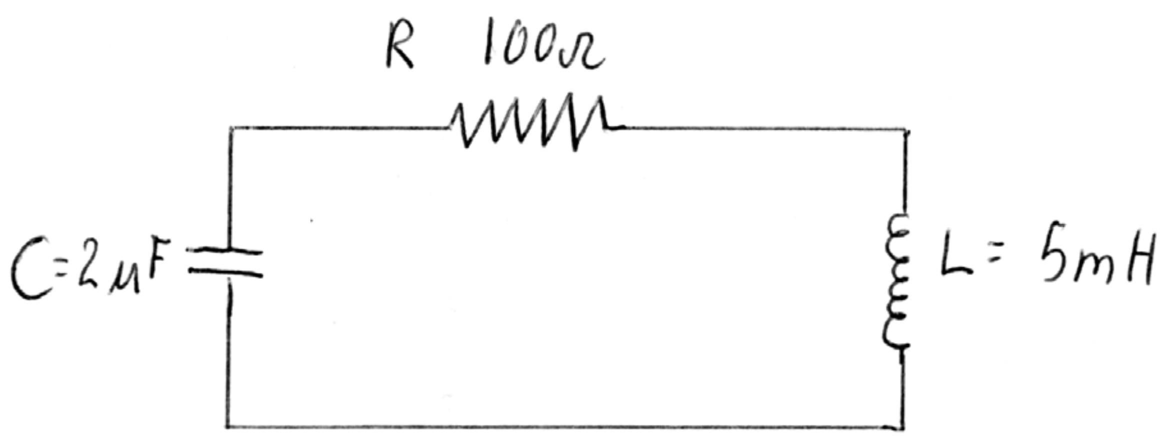


Critical Damped RLC Two Solutions

- The critically damped equation has two solutiosn

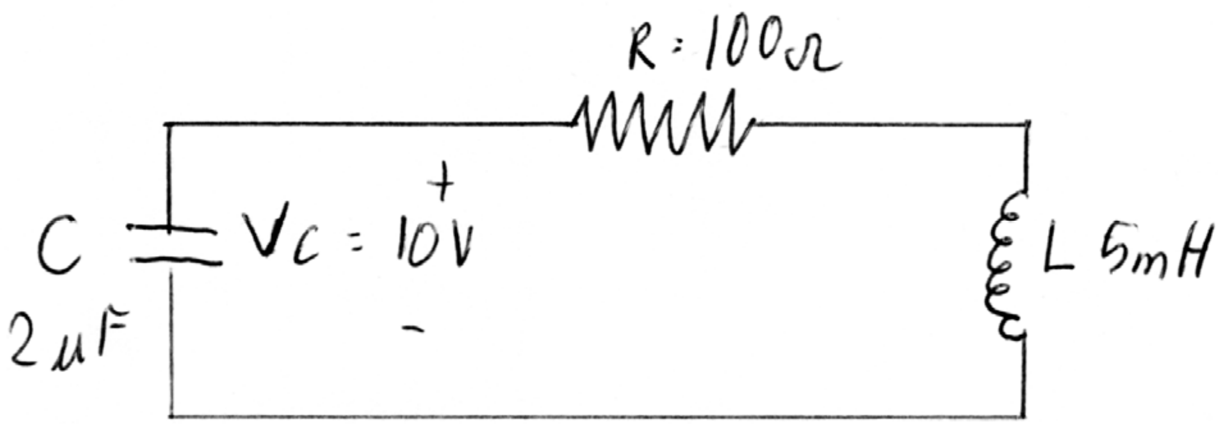
$$i(t) = [A_1 + A_2 t] \exp\left(-\frac{Rt}{2L}\right)$$

- Solution has two possible behaviours depending on A values
- The characteristic equation has two identical solutions
- Get only one oscillation (transaction)
- Or get slow approach to final value
- Difference depends on initial conditions only
- If starts with i(t=0) = 0, slow approach to rest value
- If start with i(t=0) ≠ 0, get one oscillation then rest value
- Thus same circuit will have different solutions
- Depending on the initial current conditions & energy storage



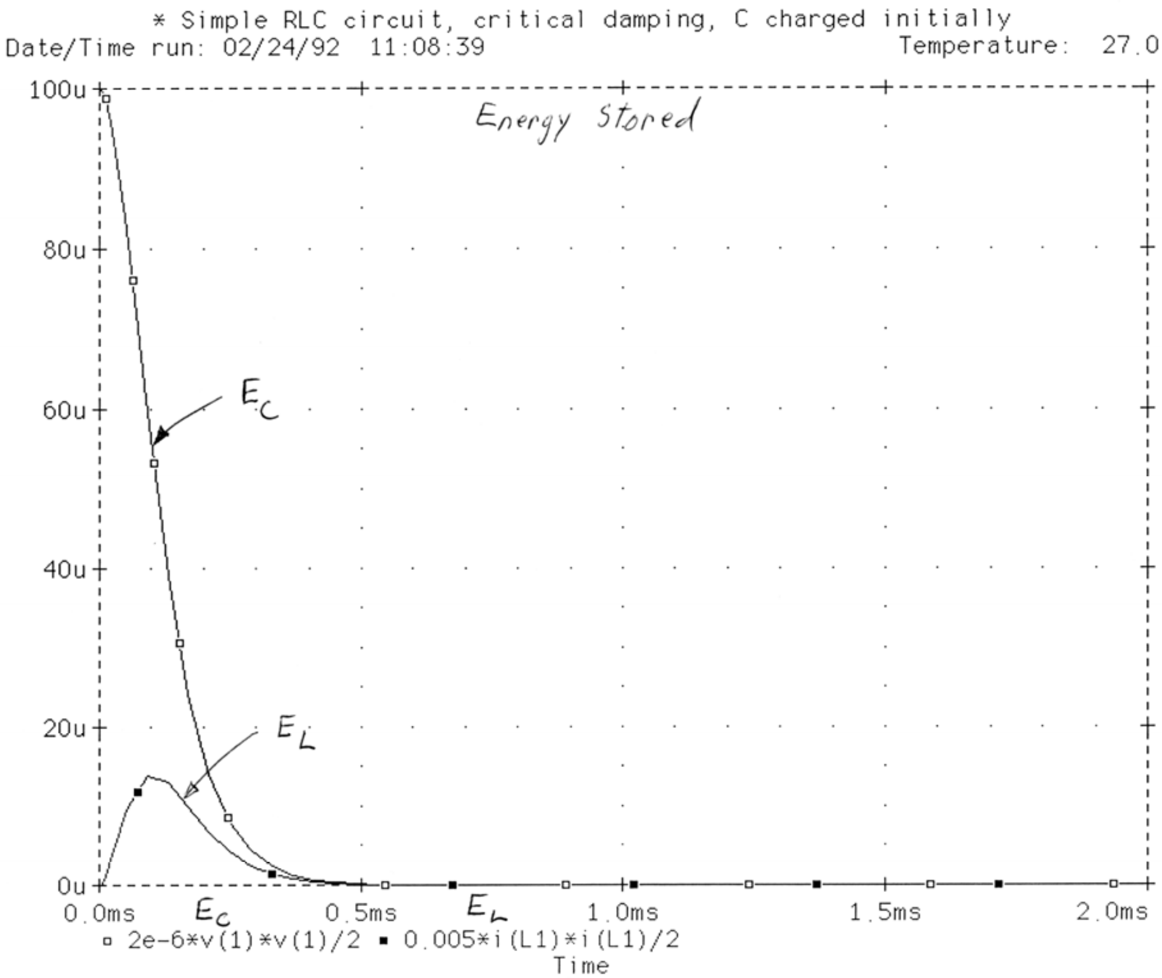
Critical Damped RLC Example i(0)=0

- Keeping L & C the same
- If R is increased 100 ohm get critical damping
- Here L = 5 mH, C = 2 μF, R = 100 Ω
- Also assume C is charged to 10 V at t = 0
- Also assume C is charged to 10 V at t = 0 but i(t)=0
- This is the no oscillation case



Critical Damped RLC, i(t=0) = 0 Energy Plot

- Energy is transferred from Capacitor to Inductor
- Inductor energy is later and higher than overdamped case
- Then both energies decline



Critical Damped RLC i(0)=0 Solution

- We define the **Damping Decay Constant** α

$$\alpha = \frac{R}{2L} = \frac{100}{2 \times 0.005} = 10^4 \sec^{-1}$$

- The damping constant gives how fast energy is decaying
- The basic Critical Damping equation is

$$i(t) = [A_1 + A_2 t]\exp\left(-\frac{Rt}{2L}\right)$$

- Solving for A constants from the initial conditions
- Since current is at t=0 is zero then

$$i(t=0) = 0 = [A_1 + A_2 0]\exp\left(-\frac{R0}{2L}\right) = A_1$$

$$i(t) = A_2 t \exp\left(-\frac{Rt}{2L}\right)$$

- Again since the inductor acts open at time zero

$$\frac{di(0)}{dt} = \frac{V_c(0)}{L}$$

- Now applying this to the equation

$$\frac{di(t=0)}{dt} = A_2 \exp\left(-\frac{Rt}{2L}\right)\left[1 - \frac{Rt}{2L}\right] = A_2 \exp\left(-\frac{R0}{2L}\right)\left[1 - \frac{R0}{2L}\right] = A_2$$

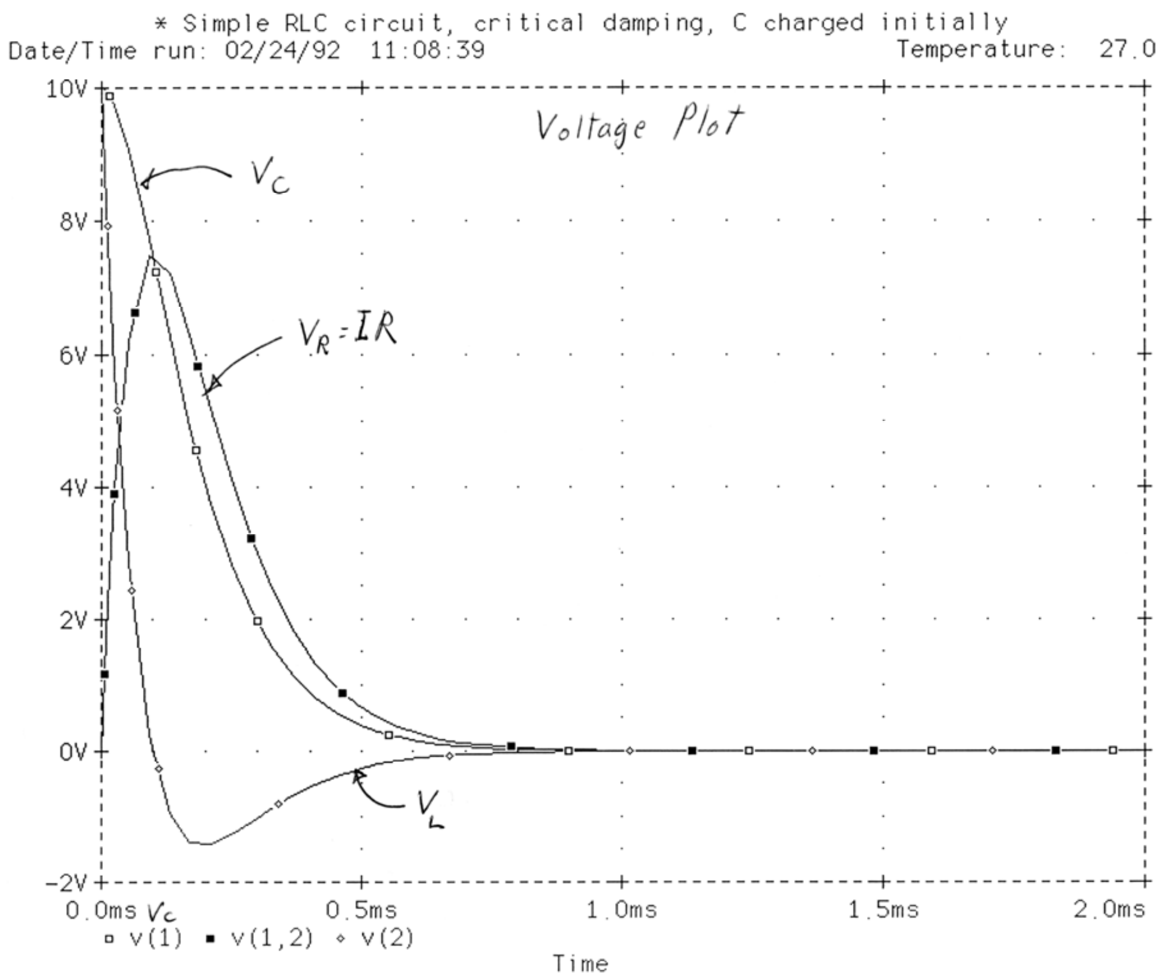
- Thus at time t=0 then

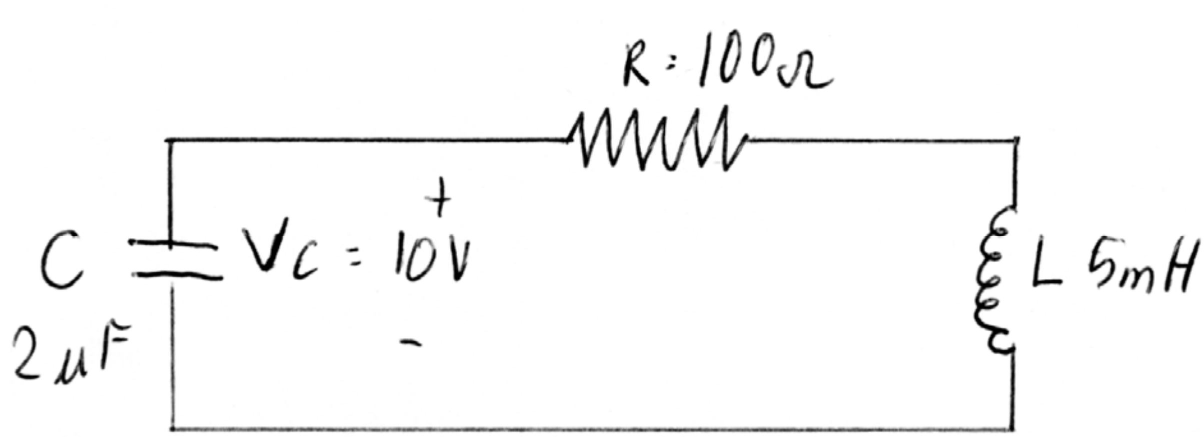
$$A_2 = \frac{V_c(0)}{L} = \frac{10}{0.005} = 2000\ A$$

$$i(t) = [A_2 t]\,exp\left(-\frac{Rt}{2L}\right) = 2000\ t\ exp\left(-10^4 t\right) A$$

Critical Damped RLC, i(t=0) = 0 Voltage Plot

- Capacitor voltage starts at 10 V and declines
- Inductor voltage starts at 10 V then reverses & declines
- Resistor voltage starts at zero, rises to peak above V_c then declines

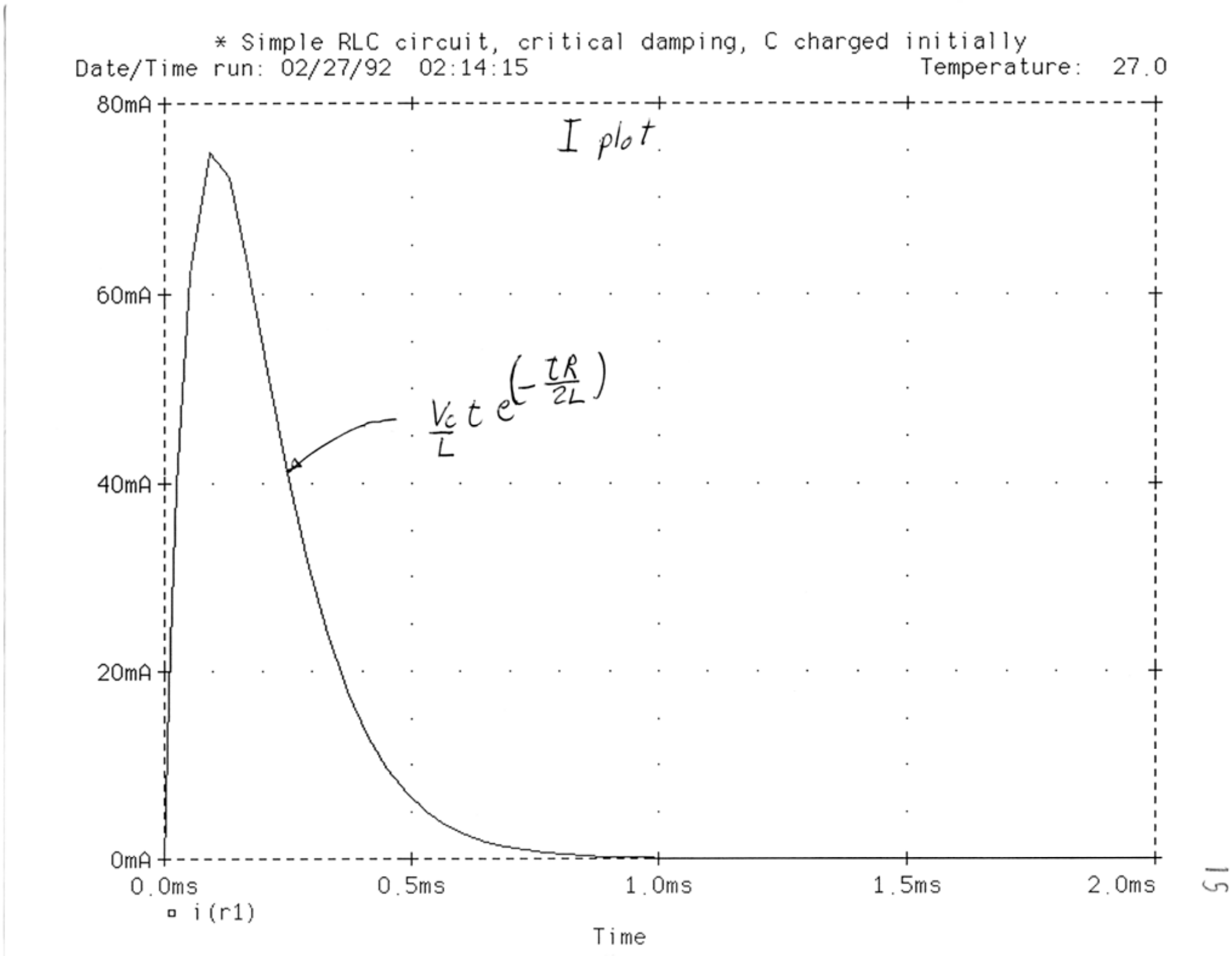




Critical Damped RLC, i(t=0) = 0 Current Plot

- Current rises to peak due to A_2t term
- Then current declines near exponential with α

$$i(t) = [A_2 t]\, exp\left(-\frac{Rt}{2L}\right) = 2000\ t\ exp\left(-10^4 t\right) A$$



Critical Damped RLC, i(t=0) ≠ 0

- Now consider the case when i(t=0) is non zero
- The practical case is when capacitor is uncharged
- But inductor has current flowing in it
- The equation has both constants nonzero

$$i(t) = [A_1 + A_2 t]\exp\left(-\frac{Rt}{2L}\right)$$

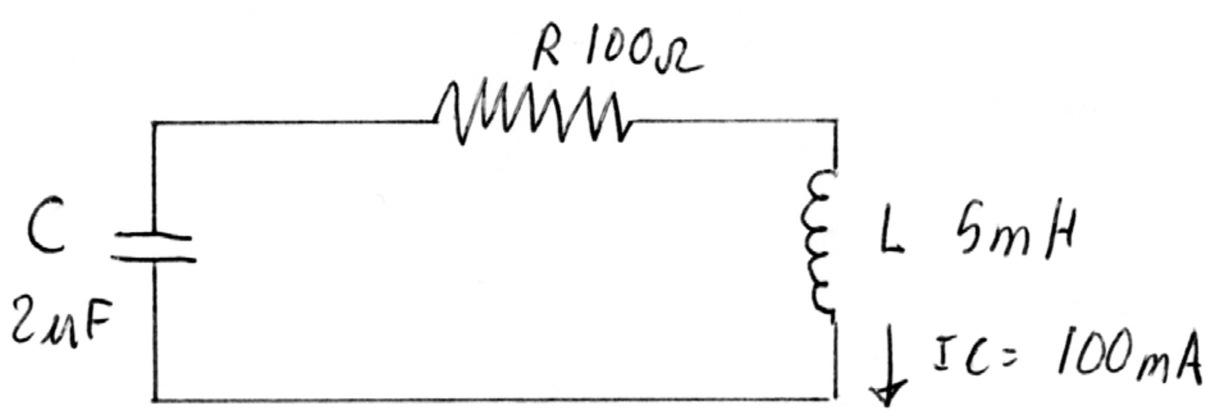
- Again the example have L = 5 mH, C = 2 μF, R = 100 Ω
- Now assume L initially carries 100 mA at t = 0, thus

$$i(t=0) = I_0 = A_1 = 100\ mA$$

- Since C acts as a short at time t=0 thus $V_C(t=0) = 0$
- Then the only voltage drop is across the resistance

$$I_0 R + L\frac{di(t=0)}{dt} = 0$$

$$\frac{di(t=0)}{dt} = -\frac{I_0 R}{L}$$



Critical Damped RLC, i(t=0) ≠ 0 Equation

- Now for the derivative of the equation

$$\frac{di(t)}{dt} = \left\{ A_2 + [A_1 + A_2 t]\left(-\frac{R}{2L}\right)\right\} \exp\left(-\frac{Rt}{2L}\right)$$

- For the initial conditions

$$\frac{di(t=0)}{dt} = \left\{ A_2 + [A_1]\left(-\frac{R}{2L}\right)\right\} \exp\left(-\frac{R0}{2L}\right) = A_2 - A_1\left(\frac{R}{2L}\right) = A_2 - \frac{RI_0}{2L}$$

- Relating this to the resistance

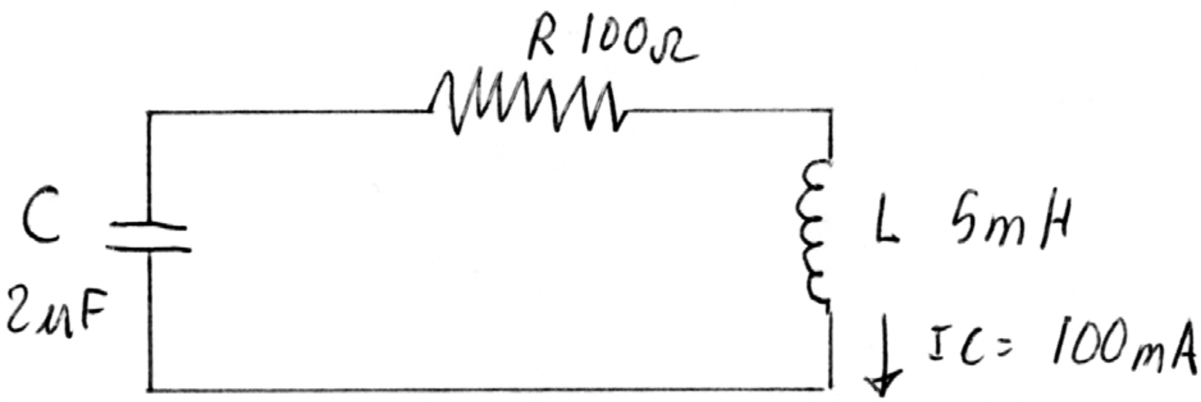
$$\frac{di(t=0)}{dt} = A_2 - \frac{RI_0}{2L} = -\frac{I_0 R}{L}$$

$$A_2 = \frac{RI_0}{2L} = -\frac{100 \times 0.1}{2 \times 0.005} = -1000\ A$$

- Thus the critically damped i(t=0)≠0 current equation is

$$i(t) = [A_1 + A_2 t]\, exp\left(-\frac{Rt}{2L}\right) = [0.1 - 1000\ t]\, exp\left(-10^4\, t\right) A$$

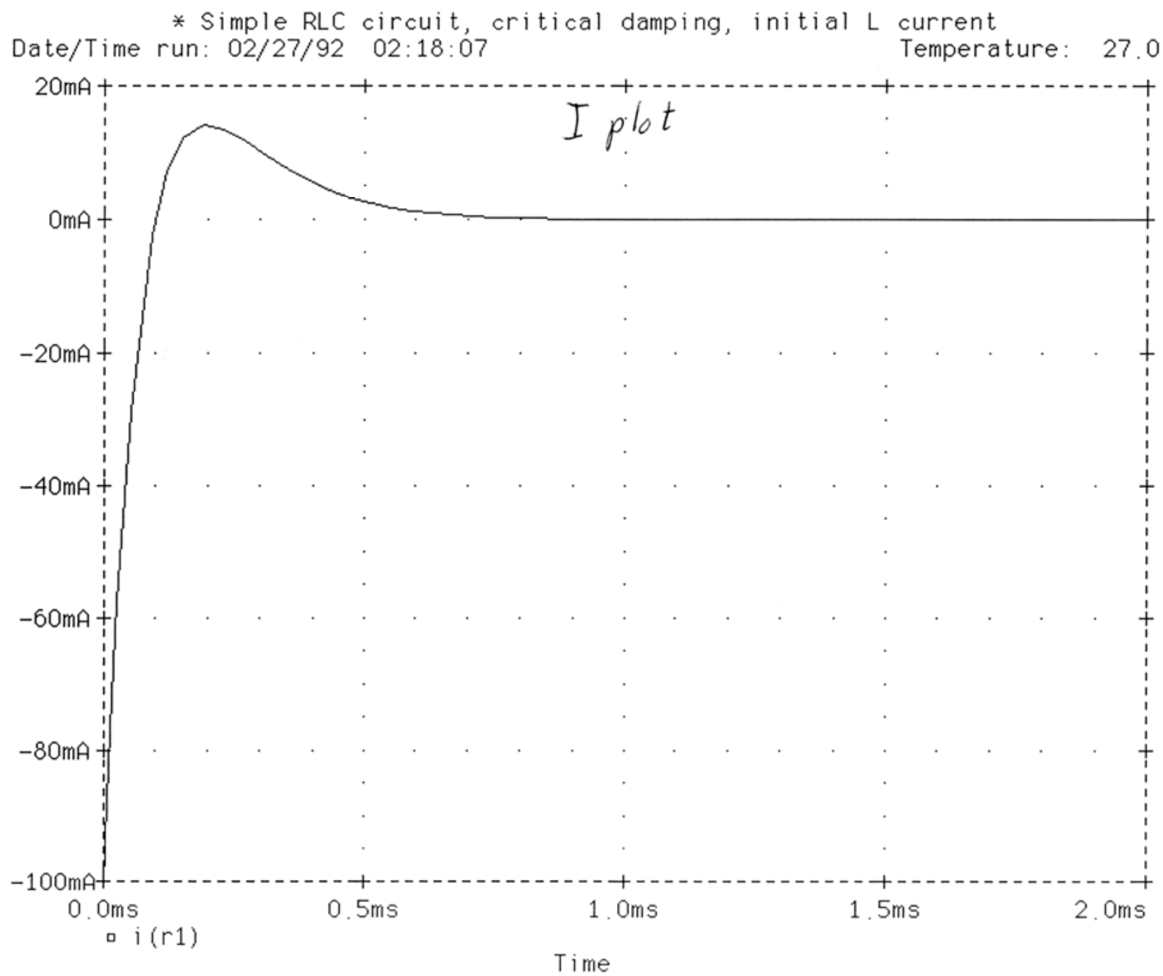
- Thus the current will reverse direction at t=0.1 msec.



Critical Damped RLC, i(t=0) ≠ 0 Current Plot

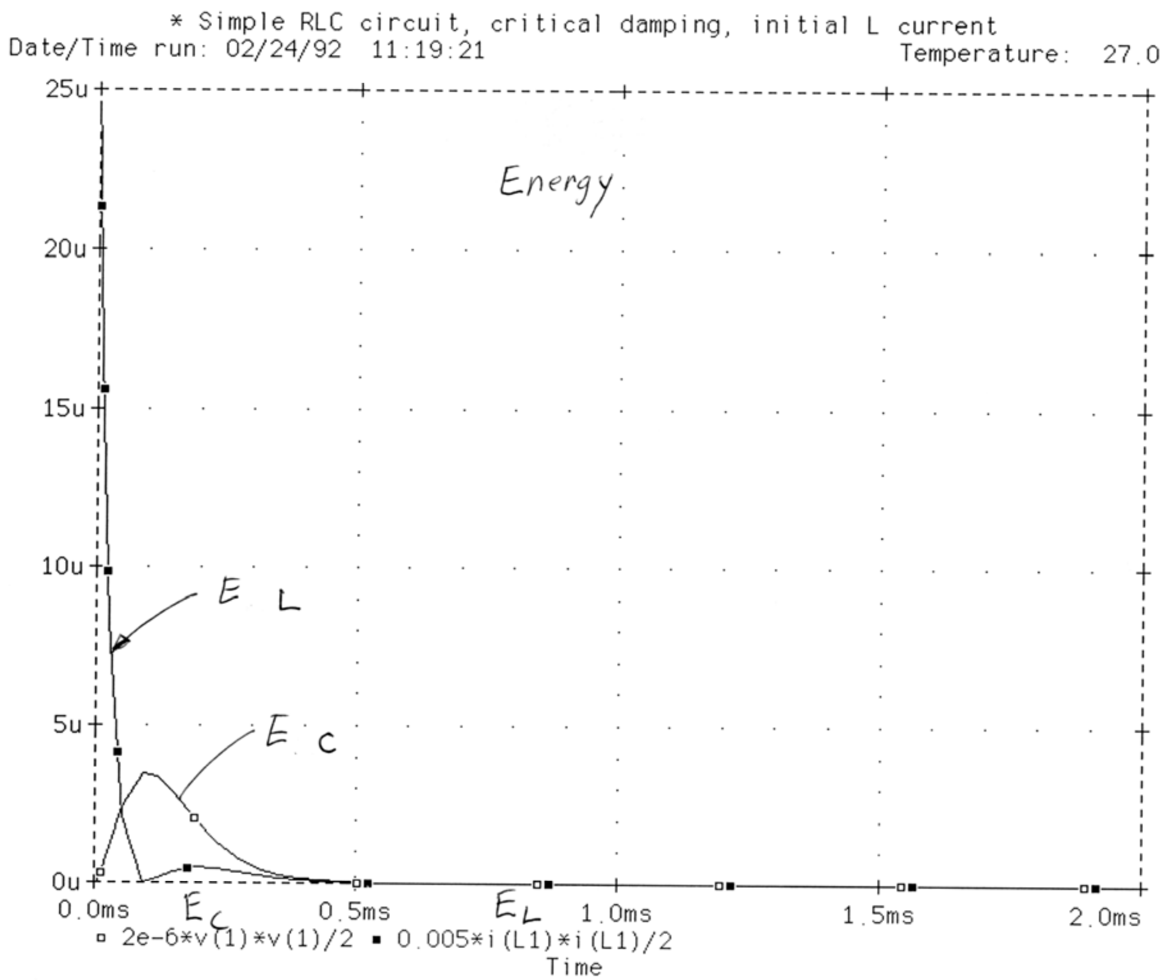
- Current reaches zero at t=0.1 msec
- Reverses, reaches peak at t=0.2 msec then declines

$$i(t) = [A_1 + A_2 t] exp\left(-\frac{Rt}{2L}\right) = [0.1 - 1000\ t] exp\left(-10^4 t\right) A$$



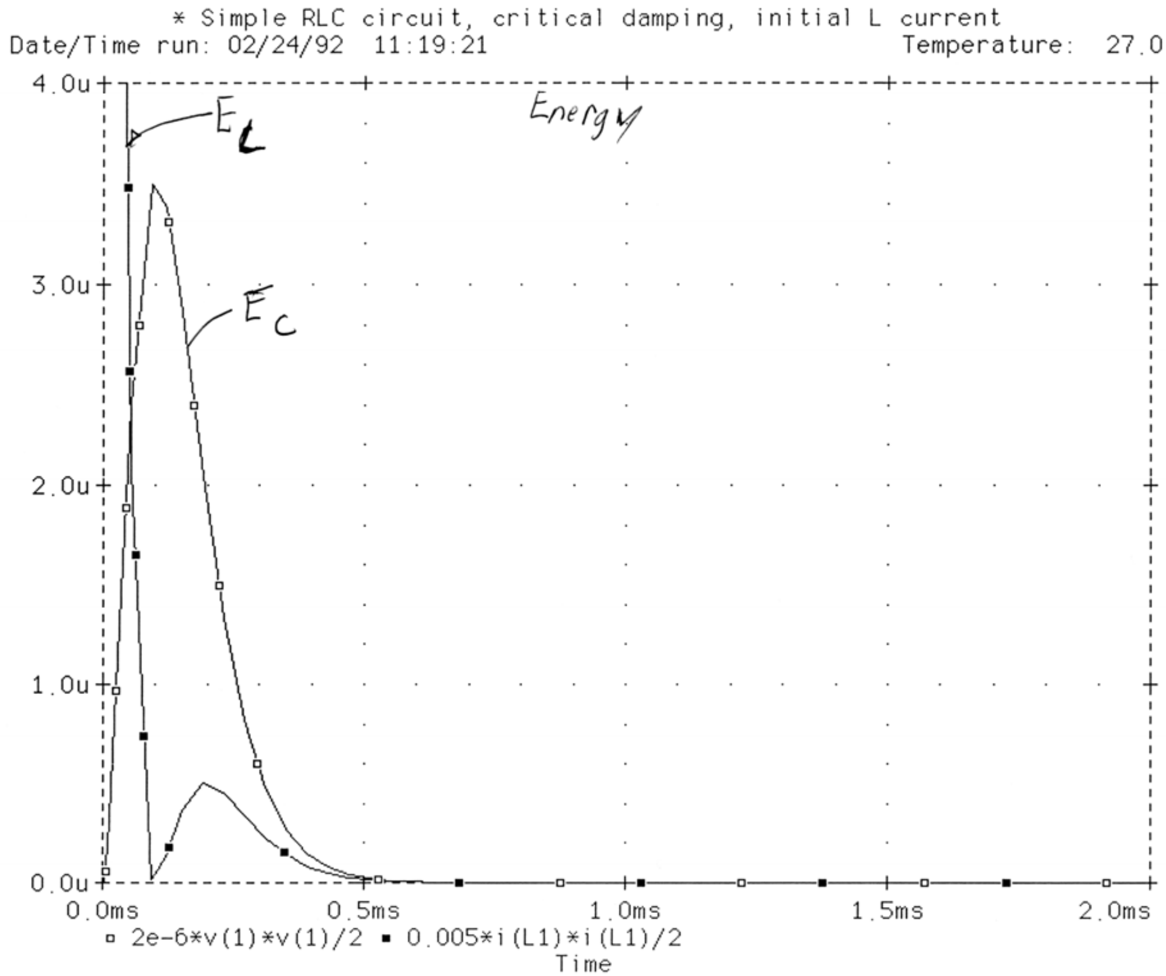
Critical Damped RLC, i(t=0) ≠ 0 Energy Plot

- Inductor energy falls to minimum at t=0.1 msec
- Energy is transferred from L to C and back to L



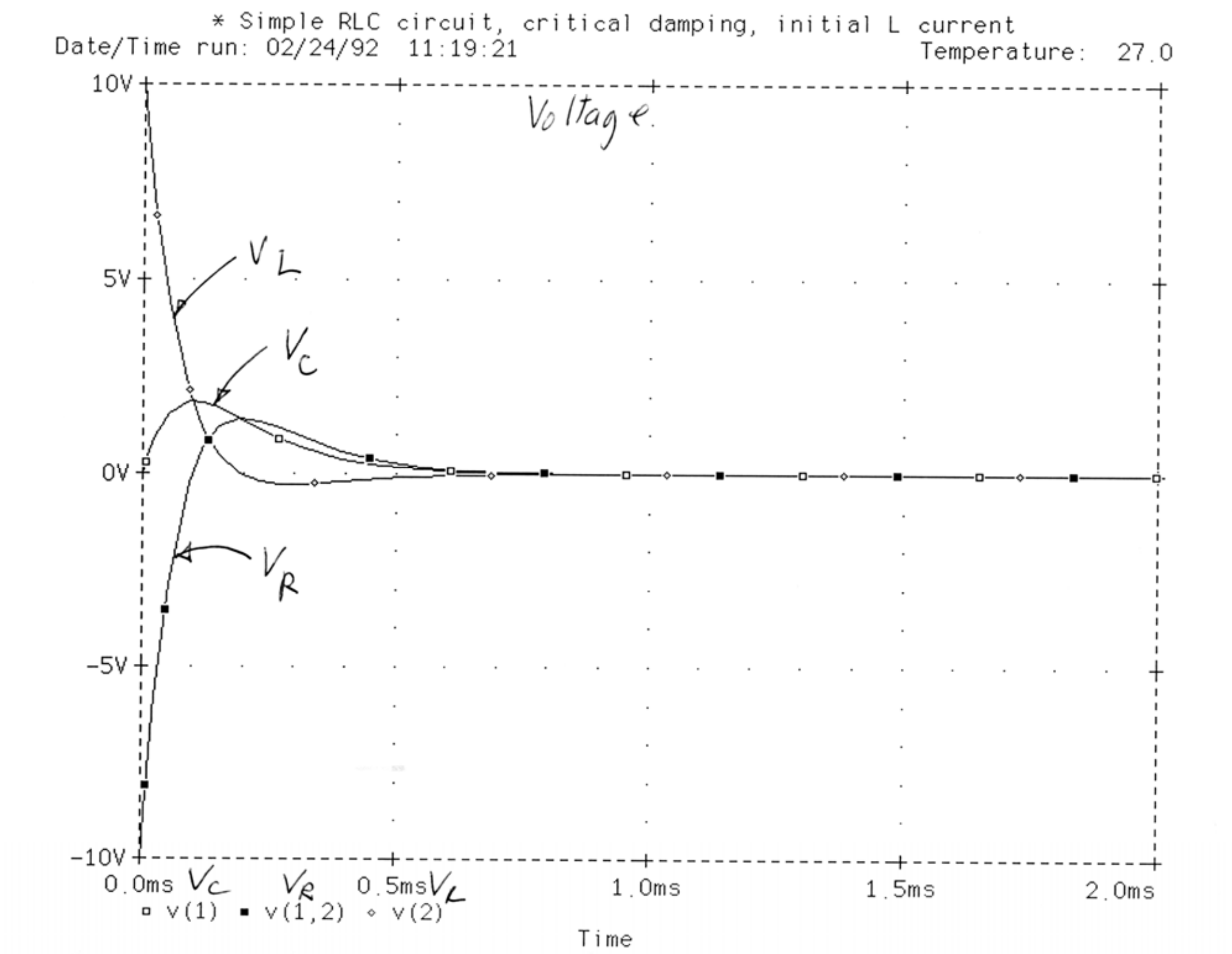
Critical Damped RLC, $i(t=0) \neq 0$ Energy Plot Expanded

- Capacitor energy reaches max when E_L is minimum
- Then Inductor energy rises again
- Both energies decay



Critical Damped RLC, i(t=0) ≠ 0 Voltage Plot

- Inductor voltage starts at 10 V and declines
- Resistor voltage starts at -10 V and rises to zero



Critical Damped RLC, i(t=0) ≠ 0 Voltage Plot Expanded

- Resistor voltage reaches zero at t=0.1 msec
- Capacitor voltage reaches max also at t=0.1 msec
- Inductor voltage falls to zero at t=0.2 msec then goes negative
- Reason V_L depends on direction of Current derivative
- Resistor voltage changes to positive, reaches peak then declines