Functional Transforms

- Often transfer impedance circuit to "Frequency Domain"
- Also called the "s" domain
- Mathemtically this is done by Laplace Transform Operator

$$L[f(t)] = \int_0^{\infty} f(t) exp(-st) dt = F(s)$$

- NOTE: traditionally tranformed function is capitalized and give as a function of s
- Then can define an inverse tranform operator such that

$$L^{-1}[F(s)] = f(t)$$

Laplace Transforms

- for a step function:

$$L[f(t)] = \int_0^\infty U_{-1}(t)exp(-st)dt = \int_0^\infty 1exp(-st)dt = \frac{1}{s}$$

- Generally do not integrate
- just look up in transform table

TABLE 15.1

AN ABBREVIATED LIST OF LAPLACE TRANSFORM PAIRS

$f(t)(t > 0^-)$	TYPE	$F(s)$
$\delta(t)$	(impulse)	1
$u(t)$	(step)	$\frac{1}{s}$
t	(ramp)	$\frac{1}{s^2}$
e^{-at}	(exponential)	$\frac{1}{s+a}$
$\sin \omega t$	(sine)	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	(cosine)	$\frac{s}{s^2+\omega^2}$
te^{-at}	(damped ramp)	$\frac{1}{(s+a)^2}$
$e^{-at} \sin \omega t$	(damped sine)	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	(damped cosine)	$\frac{s+a}{(s+a)^2+\omega^2}$

S domain and circuit elements

- Similarly for capacitors

$$I(s) = L\left[C\frac{dv(t)}{dt}\right] = sCV(s) - CV(0-)$$

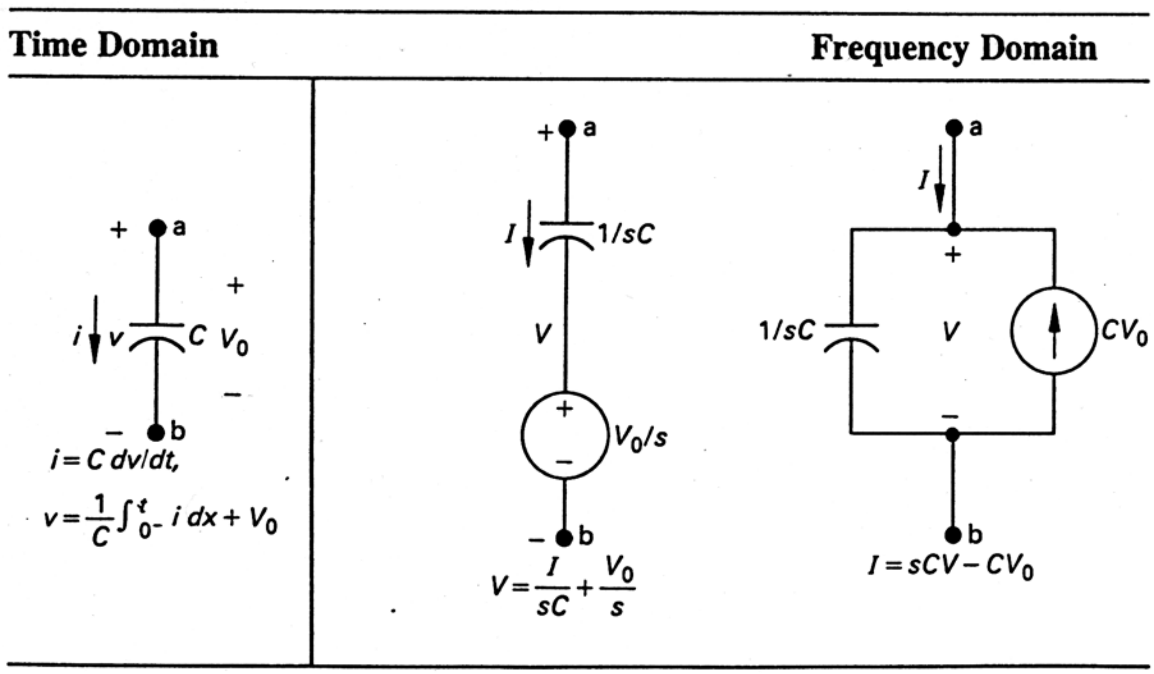
- or in terms of Voltage

$$V(s) = \frac{I(s)}{sC} + \frac{V_0}{s}$$

- which is what you expect from the impedance relationship

$$V = \frac{I}{j\omega C}$$

SUMMARY OF THE *s*-DOMAIN EQUIVALENT CIRCUITS



Transfer function: Poles and Zeros

- Recall the Transfer Characteristics:
- Relates the input signal to the output
- can define a Transfer Funcition:

$$H = \frac{V_{out}}{V_{in}}$$

- which generally will be a complex function of frequency
- Can rewrite the Transfer function in the "s" domain H(s)
- where define

$$s = j\omega$$

- If input is V, output is I then transfer is admittance

$$V = IY \qquad H(s) = Y(s)$$

- If input is I, output is V then transfer

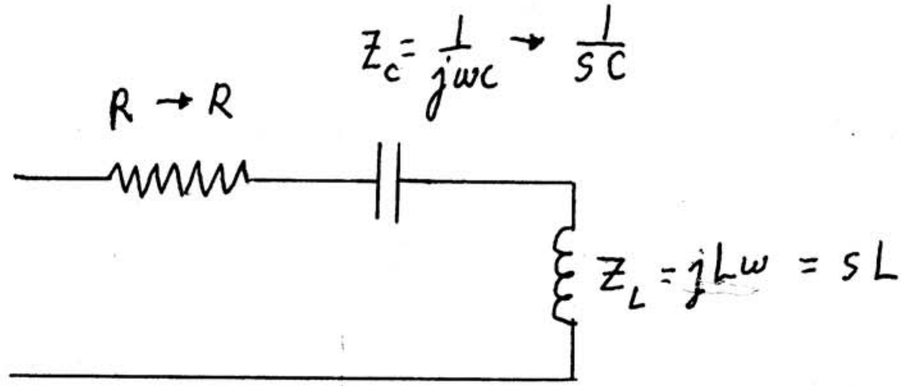
$$V = IZ \qquad H(s) = Z(s)$$

Solving Natural Response problems with S domain

- Easily solve natural response with Laplace transforms
- Basic method: using impedances for the Transfer Function
- May be Z or Y or more complex form
- Then make the S domain substitution of

$$s = j\omega$$

- Bring the Numerator and Denominator to polynomials in S
- Actually the Laplace Transform of the circuit
- But with no initial conditions
- General solution: Inverse Transform
- However for many circuits there is a short form.
- To solve we introduce Poles and Zeros concept.



Poles, Zeros of Z an Y

- Recall some frequencies give minimum impedance
- Others give infinite impedance
- In complex math get a new concept:
- zeros, complex frequency where Z(s) = 0
- poles, complex frequency where Z(s) = ∞
- then in general get Z of the form

$$Z(s) = \frac{A_0 + A_1 s + A_2 s^2 ....A_n s^n}{B_0 + B_1 s + B_2 s^2 ....B_n s^m}$$

- Solve for roots of numerator and denominator separately
- and rewrite this as

$$Z(s) = \frac{(s - s_{a1})(s - s_{a2})....(s - s_{an})}{(s - s_{b1})(s - s_{b2})....(s - s_{bm})}$$

- This gives us information about the solution

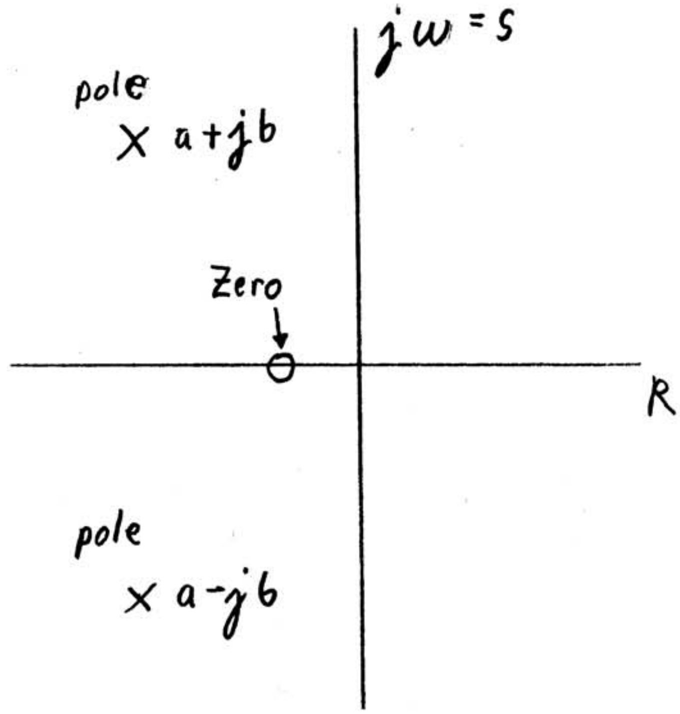
Poles, Zeros of Z an Y (EC 15.7-15.8)

$$Z(s) = \frac{(s - s_{a1})(s - s_{a2})....(s - s_{an})}{(s - s_{b1})(s - s_{b2})....(s - s_{bm})}$$

- Roots of the numerator (top) generate Zeros
- points where Z = 0
- Roots of denominator (bottom) called Poles
- points where Z becomes infinite.
- since

$$Z = \frac{1}{Y}$$

- Then poles of Z are the zeros of Y
- The zeros of Z are the poles of Y
- Plotting jω against real
- show poles as x, Zeros as o



Laplace Transform Solutions to Transfer Functions

- Consider the transfer function

$$T(s) = \frac{(s - s_1)(s - s_2)....(s - s_n)}{(s - s_{d1})(s - s_{d2})....(s - s_{dm})}$$

- Exact solution using the "Partial Fraction" method
- It can be shown that T(s) can be rewritten in the from:

$$T(s) = \sum_{j=1}^{m} \frac{K_j}{s - s_{dj}}$$

- where K_j = constants of the fraction
- assumed here that the denominator has a higher s power
- Then use the Inverse Transform relations: most commonly

$$\mathbf{L}^{-1}\left[\frac{1}{s - a}\right] = exp(-at)$$

- Hence the general solutions become

$$T(t) = \sum_{j=1}^{n} I_j exp(s_{dj} t)$$

- Note: some of these will be complex numbers

Poles, Zeros and Transfer function

- Consider that pole and zero come from transfer function

$$I = \frac{V}{Z} = VY$$

- The natural response for current of an applied voltage is

$$I = \sum_{j=1}^{n} I_j exp(s_j t)$$

- where s's are zeros of Z or poles of Y

- Similarly for Voltages

$$V = IZ = \frac{I}{Y}$$

- The natural response for voltage of an applied current is

$$V = \sum_{j=1}^{n} V_j exp(s_j t)$$

- where s's are poles of Z or zeros of Y

Using Complex Impedance to Natural Response

- The transfer function best be obtained Z or Y
- Thus can solve the behaviour with Z or Y
- Must convert to the transfer function F(s)

- Procedure for S function

(1) Write the Impedance Z or admittance Y of circuit

(2) replace the complex frequency with s

$$s = j\omega$$

(3) Bring s equation to a common denominator

(4) Zeros are in numerator, Poles in Denominator

solve the s equations to find those roots in each

(5) rewrite equation in the form

$$T(s) = \frac{(s - s_1)(s - s_2)....(s - s_n)}{(s - s_{a1})(s - s_{a2})....(s - s_{am})}$$

Example Series RLC Poles/Zeros

- For the simple RLC circuit the Z is

$$Z = R + j\omega L + \frac{1}{j\omega C}$$

- For the simple RLC circuit driven by a voltage

$$I = \frac{V}{Z} = VY$$

- converting the Z to a s function

$$Z(s) = R + sL + \frac{1}{sC}$$

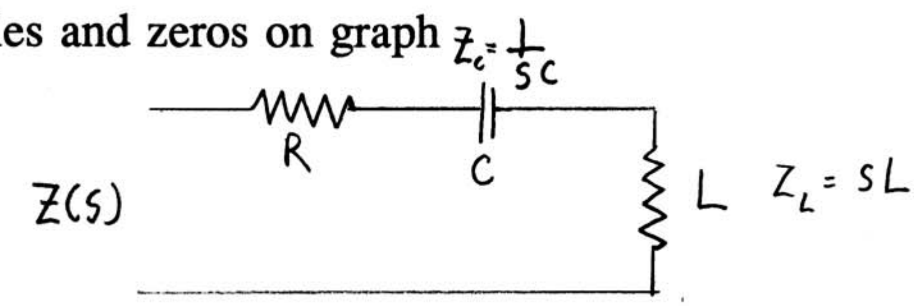
- now bring to a common denominator

$$Z(s) = \frac{L\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right]}{s}$$

- This has 2 zeros in denominator, 1 pole from numerator

$$s_{zeros} = -\frac{R}{2L} \pm \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right]^{1/2}$$

- plot poles and zeros on graph



Example Series RLC Poles/Zeros

- for the example case L = 5 mH, C = 2 μF, R = 10 ohms

$$Z(s) = \frac{L\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right]}{s} = \frac{0.005\left[s^2 + \frac{10}{0.005}s + \frac{1}{0.005x\,2x\,10^{-6}}\right]}{s}$$

- this is the same quadratic equation as before
- solving for the zeros of the quadratic as before

$$\alpha = \frac{R}{2L} = \frac{10}{2x\,0.005} = 10^3$$

- the damped frequency is:

$$\omega^2 = \omega_n^2 - \alpha^2 = 10^8 - 10^6 = 9.9x\,10^7$$

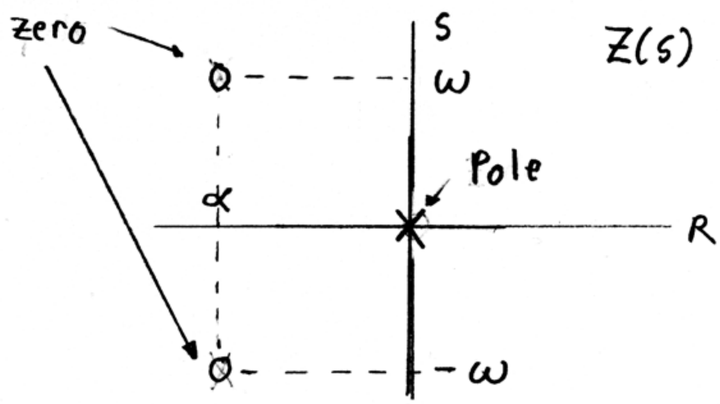
$$\omega = 9.95x\,10^3$$

- Then the solutions are

$$s_1 = -\alpha + j\omega \qquad s_2 = -\alpha - j\omega$$

- Using these Zeros combined natural response is

$$i(t) = A_1\exp([-\alpha + j\omega]t) + A_2\exp([-\alpha - j\omega]t)$$



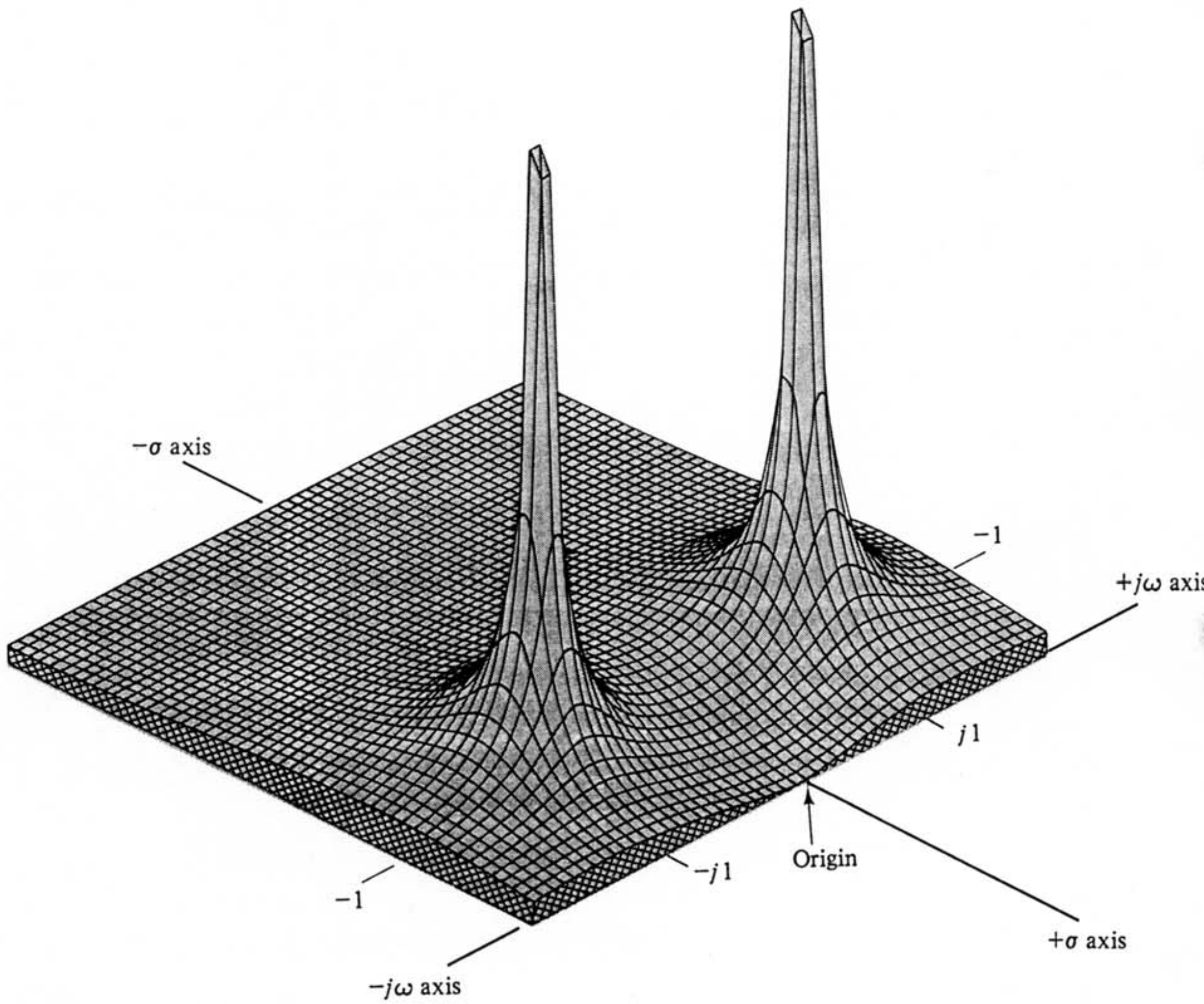


Figure 10-4.1 Magnitude function representation on the complex plane.

Poles/Zeros locations and response type

- The location of the poles determines the response type
- for stable systems all responses on left hand side

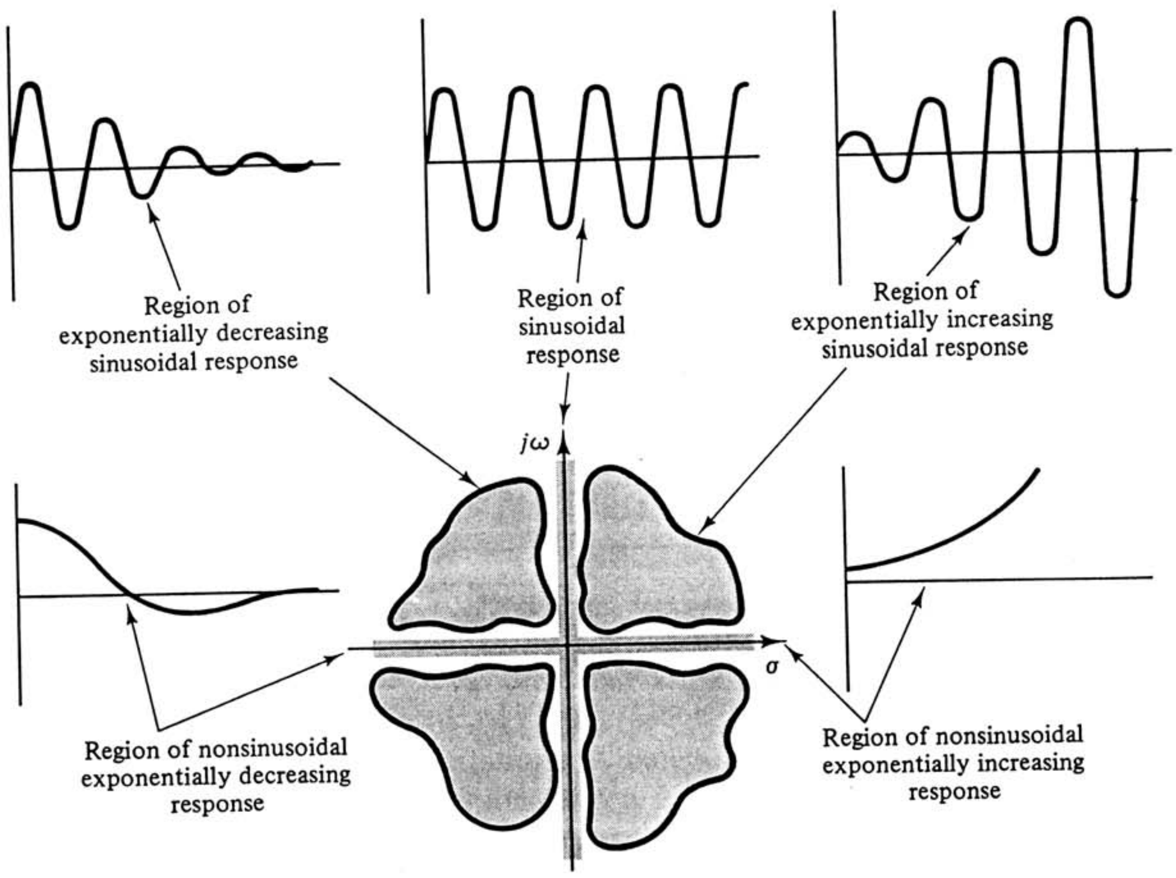


Figure 10-4.2 Impulse response characteristics.

Transfer Functions and Frequency Response

- Consider simple RC circuit
- Looking across C the voltage divider is

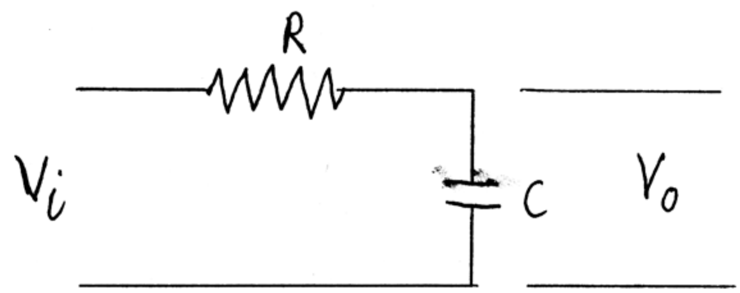
$$V_o = \frac{Z_C}{Z_C + Z_R} V_i = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}$$

Thus the Transfer function is

$$\frac{V_o}{V_i} = \frac{1}{1 + j\omega RC}$$

- in substituting for s in the Transfer function

$$T(s) = \frac{1}{1 + sRC} \qquad s = j\omega$$



Transfer Functions and Frequency Response (

- For high frequencies

$$\frac{V_o}{V_i} \rightarrow \frac{1}{j\omega RC} \qquad \theta_T \rightarrow 90^o$$

- Thus output decreases by 10 per 10 increase in ω

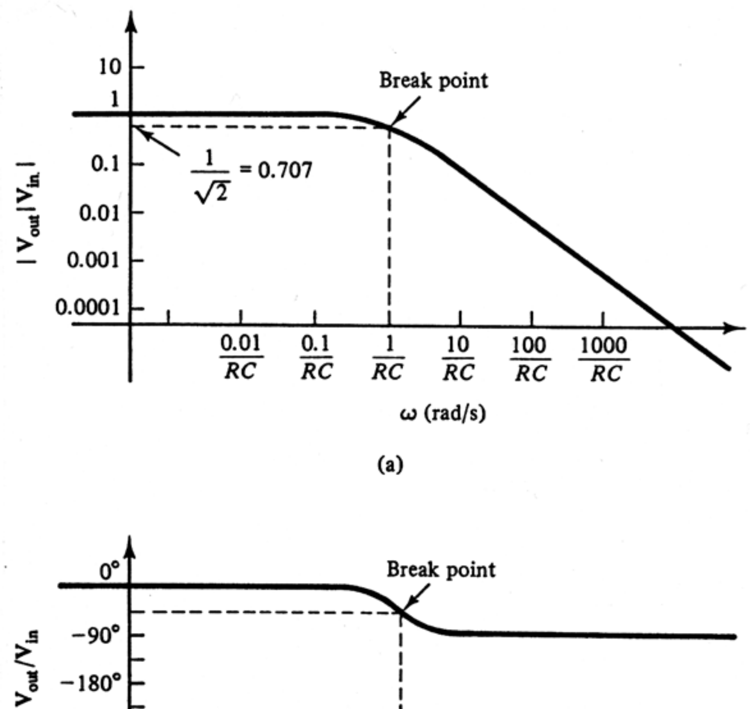
- At the cutoff or "Break Point" frequency

Real = Imaginary

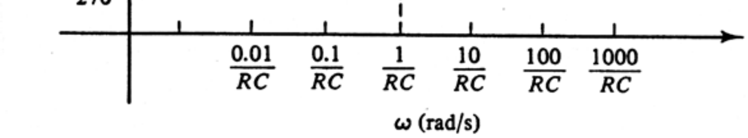
$$\omega_0 = \frac{1}{RC}$$

$$\left|\frac{V_o}{V_i}\right| = \frac{1}{\sqrt{2}} \qquad \theta_{\omega_0} = -45^o$$

- Plot Transfer function on log log plot
- Called Bode Plot



(a)



(b)

Bode Plots and Frequency Response

- Measure the rate of fall off in terms power loss
- Unite used is the Decibel (dB)

$$dB = 10 \log_{10}\left[\frac{P_{out}}{P_{in}}\right]$$

- since

$$P = \frac{V^2}{Z}$$

- the standard dB formula is

$$dB = 20 \log_{10}\left[\left|\frac{V_o}{V_i}\right|\right]$$

- Any circuit with transfer function like

$$T(s) = \frac{1}{1 + sRC}$$

- output drops at 20 dB per decade (10x) frequency
- also express this as 6 dB per octave
- one Octave is a doubling of the frequency

- at the breakpoint output is down by

$$dB = 20 \log_{10}\left[\frac{1}{\sqrt{2}}\right] = -3dB$$

- thus know Bode plot just from Transfer function

Bode Plot of RC with R output

- Now consider RC with output across R

$$\frac{V_o}{V_{in}} = \frac{Z_R}{Z_C + Z_R} = \frac{R}{R + \dfrac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC}$$

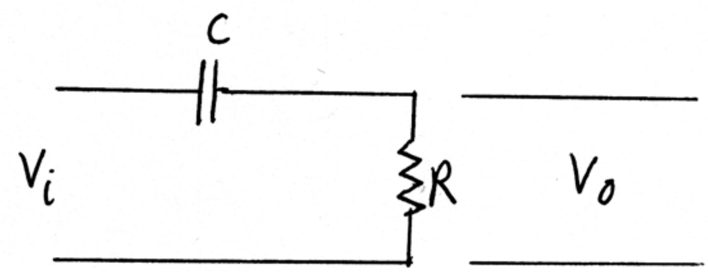
$$T(s) = \frac{sRC}{1 + sRC}$$

- Note the critical point responses

$$\omega \ll \frac{1}{RC} \quad |T(s)| \to \omega RC \quad \theta_T \to +90^o$$

$$\omega_0 = \frac{1}{RC} \quad |T(s)| \to \frac{1}{\sqrt{2}} \quad \theta_T \to +45^o$$

$$\omega \gg \frac{1}{RC} \quad |T(s)| \to 1 \quad \theta_T \to 0^o$$



Bode Plot of RC with R output

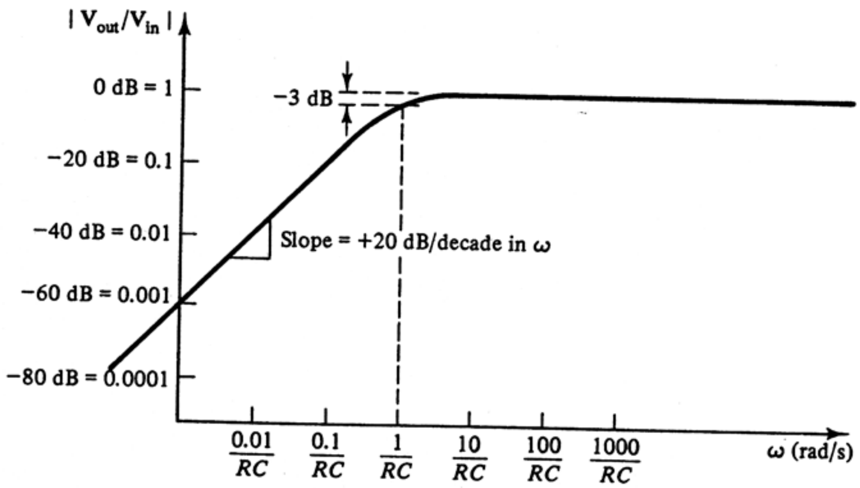
- Getting Bode Plot from Transfer Function

(1) Terms on Numerator of T(s) generate rising slope
- slope is +20 db per decade above the critical freq.
- starts at the critical frequency ω_0
- Tends to shift phase +90 degrees

(1) Terms on Denominator of T(s) generate falling slope
- slope is -20 db per decade above the critical freq.
- starts at the critical frequency ω_0
- Tends to shift phase -90 degrees

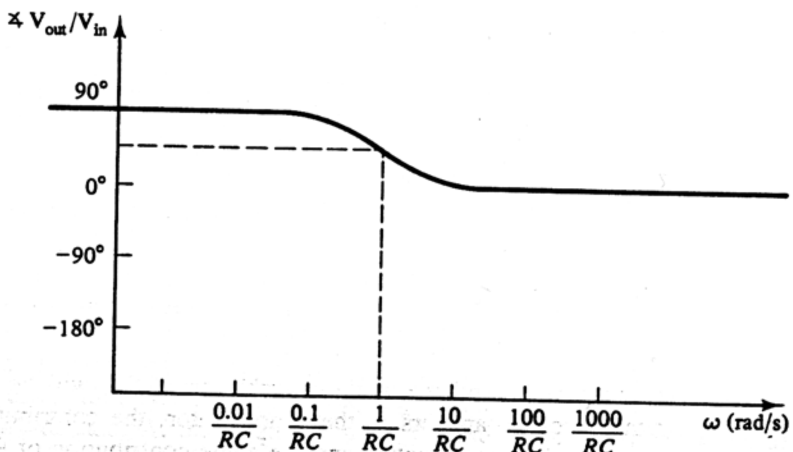
- for this example

$$T(s) = \frac{sRC}{1 + sRC}$$



| Vout/Vin |
0 dB = 1
−3 dB
−20 dB = 0.1
−40 dB = 0.01
Slope = +20 dB/decade in ω
−60 dB = 0.001
−80 dB = 0.0001
0.01/RC
0.1/RC
1/RC
10/RC
100/RC
1000/RC
ω (rad/s)

(a)



∡ Vout/Vin
90°
0°
−90°
−180°
0.01/RC
0.1/RC
1/RC
10/RC
100/RC
1000/RC
ω (rad/s)

(b)

Transfer Functions and Bode Plots (

- Consider the following T(s)

$$T(s) = 50\frac{s\left[1 + \frac{s}{10}\right]}{\left[1 + \frac{s}{10^4}\right]\left[1 + \frac{s}{10^7}\right]}$$

- Denominator 1st term produces Bode initial slope +20 db
- 2nd Denominator term adds another +20 db at 10 rad/s
- 1st Numerator reduces slope by -20 dB at 10^4 rad/s
- 2nd Numerator reduces slope by -20 dB at 10^7 rad/s

Figure 9.6
Magnitude Bode plot of the system function of Eq. (9.43).

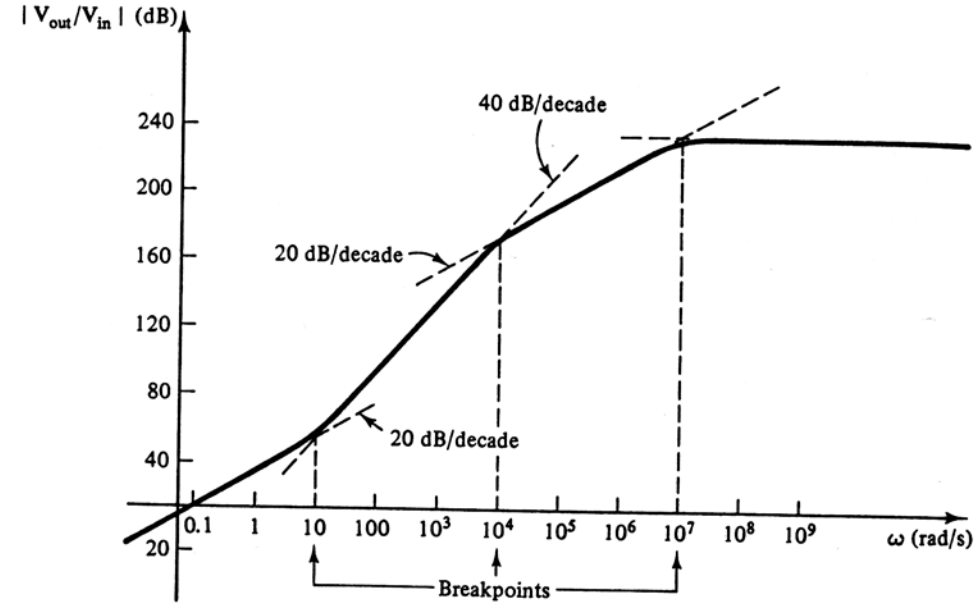
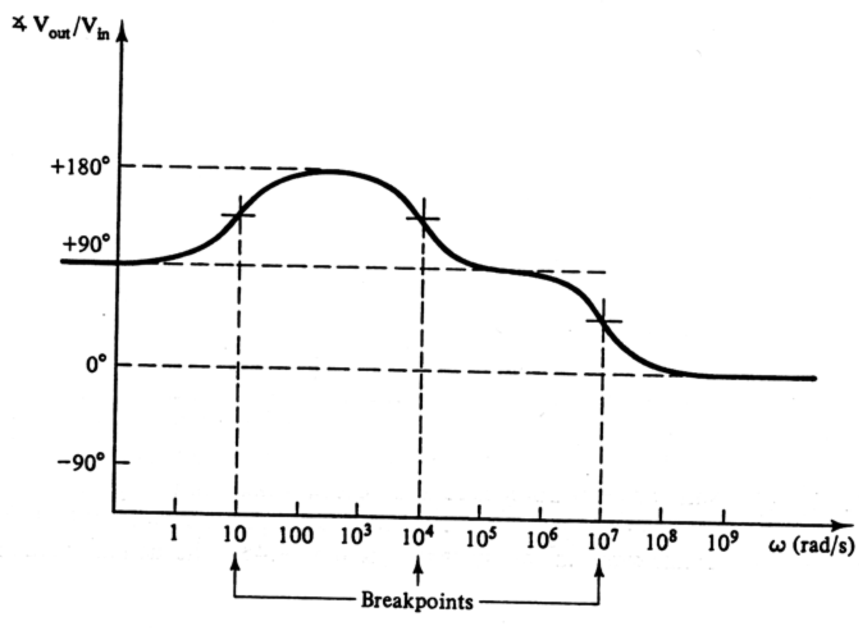


Figure 9.7
Angle Bode plot of the system function of Eq. (9.43).



Transfer Functions and Frequency Response

- When terms act at same frequency slope increased
- Example of a steeper slope

$$T(s) = \frac{100}{\left[1 + \frac{s}{10^4}\right]\left[1 + \frac{s}{10^4}\right]}$$

- get slope of +40 dB per decade for > 10^4 rad/s
- Break point is -6 dB down from flat
- Phase shift is -180 degrees

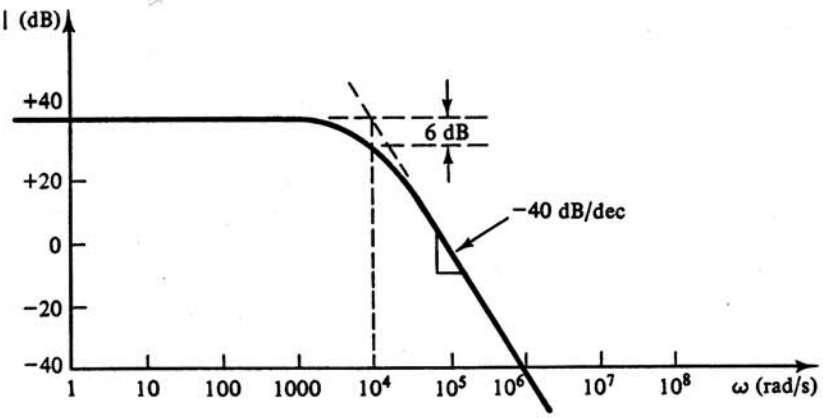


Figure 9.8 Magnitude Bode plot of the system function of Eq. (9.45).

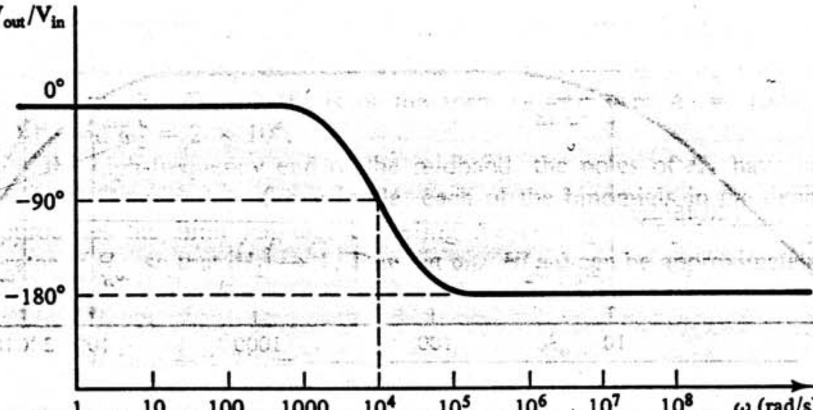


Figure 9.9 Angle Bode plot of the system function of Eq. (9.45).