ENSC327/328
Communications Systems
15: Correlation and Spectral Density

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Outline

- Review: energy signals & power signals
- 2.8 Energy spectral density and autocorrelation
  - For deterministic energy signals
- 2.9 Power spectral density and autocorrelation
  - For deterministic power signals

- In chapter 8, we generalize these definitions to random processes.

Recall: Energy Signals vs Power Signals

- Power and energy of arbitrary signal x(t):
  - Energy: $E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$
  - Power: $P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$

For periodic signals: $P = \frac{1}{T} \int_{t_0}^{t_0+T} |x(t)|^2 dt$

- Energy Signal: energy is finite

$$0 < E < \infty \quad (\text{so } P = 0)$$

- Power Signal: power is finite

$$0 < P < \infty \quad (\text{so E} = \infty)$$

Recall: Properties of FT

- Convolution Theorem:

$$\int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau \quad \leftrightarrow \quad G_1(f) G_2(f)$$

- Correlation Theorem:

$$\int_{-\infty}^{\infty} g_1(t) g_2^*(t-\tau) dt \quad \leftrightarrow \quad G_1(f) G_2^*(f)$$

- Correlation measures the similarity between g1 and shifted g2.
- Difference from convolution: no time reversal.

Rayleigh's Energy Theorem (Parseval's Theorem)

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

(Property 14 on Page 38)

Proof (see Lecture 2):

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} g(t) g^*(t) dt = \int_{-\infty}^{\infty} g(t) \int_{-\infty}^{\infty} G^*(f) e^{-j2\pi ft} df\, dt$$

$$= \int_{-\infty}^{\infty} G^*(f) \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt df = \int_{-\infty}^{\infty} G^*(f) G(f) df = \int_{-\infty}^{\infty} |G(f)|^2 df$$

$\int_{-\infty}^{\infty} |g(t)|^2 dt$ is the energy of the signal.

⇨ $|G(f)|^2$ can be viewed as energy density in freq domain.
Note that $|G(f)|^2$ has real value.

Rayleigh's Energy Theorem (Parseval's Theorem)

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$ (Property 14 on Page 38)

Alternative Proof:

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2.8 Correlation and Spectral Density for Energy Signals

- Autocorrelation function of an energy signal:

$$R_x(\tau) \overset{\Delta}{=}$$

- This is a measure of the similarity between x(t) and its shifted version x(t- τ).
- Error in the book: dτ in (2.124) should be dt.
- This definition is for deterministic signal x(t).
- In Chapter 8, we will define the autocorrelation of a random process using statistical expectation.

2.8 Correlation and Spectral Density for Energy Signals

- If the time lag τ = 0:

- From Rayleigh's Energy Theorem:

We can define $|X(f)|^2$ as the energy spectral density or energy density spectrum of an energy signal.

$$\psi_x(f) \overset{\Delta}{=} |X(f)|^2$$ Real value, Unit: (Joules/Hz)

Wiener-Khintchine Theorem

- From Correlation Theorem: $\int_{-\infty}^{\infty} g_1(t) g_2^*(t-\tau) dt \quad \leftrightarrow \quad G_1(f) G_2^*(f)$

$\Rightarrow$

ie, $Rx(\tau)$ and $|X(f)|^2$ form a Fourier transform pair!

Wiener-Khintchine Theorem

$$\psi_x(0) = \int_{-\infty}^{\infty} R_x(\tau)d\tau \qquad R_x(0) = \int_{-\infty}^{\infty} \psi_x(f)df$$

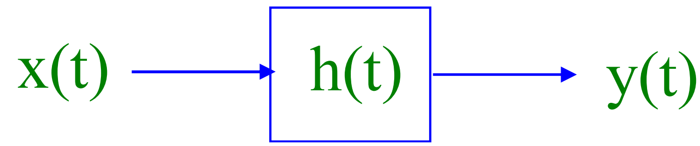
Example

- Find the autocorrelation of the sinc pulse:

$$x(t) = A\,\text{sinc}\,(t)$$

Solution:

Effect of Filtering



- If x(t) is filtered by h(t):

$$\psi_y(f) = \left|H(f)\right|^2 \psi_x(f)$$

Proof:

Cross-correlation of energy signals

- The cross-correlation between two energy signals:

$$R_{xy}(\tau) \overset{\Delta}{=}$$

- We can also define:

$$R_{yx}(\tau) \overset{\Delta}{=}$$

- Relationship: $R_{xy}(\tau) = R^{*}_{yx}(-\tau)$

Proof:

Cross-correlation of energy signals

- Orthogonal: x(t) and y(t) are orthogonal if

$$R_{xy}(0) = \int_{-\infty}^{\infty} x(t) y^*(t) dt = 0.$$

- The cross spectral density is defined as:

$\psi_{xy}(f) =$ $\psi_{yx}(f) =$

- By correlation property of FT:

$$\int_{-\infty}^{\infty} g_1(t) g_2^*(t-\tau) dt = R_{g_1 g_2}(\tau) \leftrightarrow G_1(f) G_2^*(f)$$

$\Rightarrow$

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2.9 Power spectral density

- Autocorrelation of a deterministic power signal:

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x^*(t-\tau)dt$$

- Average power of a signal: $P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$

- Fourier transform of a power signal may not exist because the energy is infinite:
  - Solution: work with the truncated signal with finite energy:

The FT of this signal exists.

Power spectral density

- The power can be written as:
- $X_T(t)$ has finite energy ➔ by Rayleigh energy theorem:

Exchanging lim and integral ➔

- So we can define power spectral density of a deterministic signal as
- Thus

Summary

- Deterministic Energy signals:

$$R_x(\tau) \overset{\Delta}{=} \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt \qquad \psi_x(f) = |X(f)|^2$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \psi_x(f)df$$

Wiener-Khintchine Theorem: $R_x(\tau) \leftrightarrow \psi_x(f)$

- Deterministic Power signals:

$$S_x(f) = \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2 \qquad P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} S_x(f)df$$

- In chapter 8, we generalize the autocorrelation, psd, and the Wiener-Khintchine Theorem to random processes.