Appendix G: Rayleigh, Rice and Chi-Squared Distributions

These notes are a heavily-adapted version of a chapter from J.K. Cavers, Mobile Channel Characteristics, **Shady Island Press**, 2004. They are intended to show how the Rayleigh, Rice, chi-squared and Nakagami-*m* probability density functions (pdfs) are related, and to introduce the Marcum Q-function.

The complex Gaussian random variables underlying all these distributions are denoted by *g*. Why? Because the discussion was originally oriented toward *gain* in a wireless fading channel.

Distributions Derived From Zero Mean Gaussian Variates

4.2.1 Rayleigh pdf

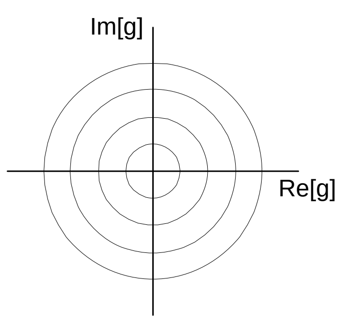
Start with a **zero-mean** complex Gaussian random variable *g* with variance σ^2 in the real and imaginary parts.

$$\sigma^2 = \frac{1}{2}\cdot E\left[\left(\left|g(t)\right|\right)^2\right] = E\left(g_I(t)^2\right) = E\left(g_Q(t)^2\right) \tag{4.2.1}$$

Note that the real and imaginary components are individually Gaussian with variance σ^2. The probability density function is then

$$p_g(g) = \frac{1}{2\cdot\pi\cdot\sigma^2}\cdot\exp\left[-\frac{1}{2}\cdot\frac{\left(\left|g\right|\right)^2}{\sigma^2}\right] \tag{4.2.2}$$

and its isoprobability contours are circles centred on the origin:



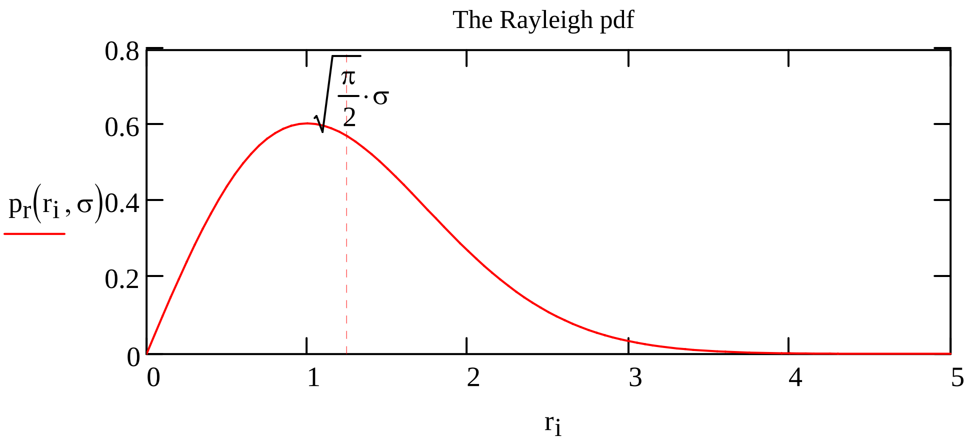
If we change to polar coordinates $g = g_I + j\cdot g_Q = r\cdot e^{j\cdot\theta}$ then standard transformations [**Papo84, Proa95, Lee82**] give the joint pdf as

$$p_{r\theta}(r,\theta) = \frac{r}{2\cdot\pi\cdot\sigma^2}\cdot\exp\left(-\frac{r^2}{2\cdot\sigma^2}\right) \tag{4.2.3}$$

Clearly, *r* and θ are independent, since the joint pdf is the product of their individual pdfs, given by

$$p_r(r,\sigma) := \frac{r}{\sigma^2}\cdot\exp\left(-\frac{r^2}{2\cdot\sigma^2}\right) \qquad r \ge 0 \quad \text{and} \quad p_\theta(\theta) = \frac{1}{2\cdot\pi} \qquad -\pi \le \theta < \pi \qquad (4.2.4)$$

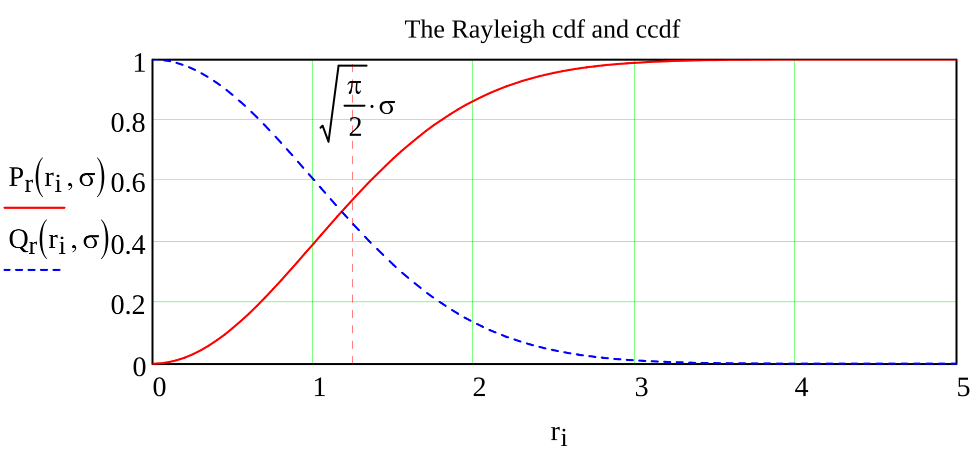
This pdf of the amplitude *r* is the *Rayleigh distribution*. Let's see how it looks, for $\sigma = 1$.



mean	mode	standard deviation
$\mu_r = \sqrt{\frac{\pi}{2}}\cdot\sigma$	σ_g	$\sigma_r = \sqrt{2-\frac{\pi}{2}}\cdot\sigma$

The cumulative distribution function and complementary cumulative distribution function are

$$P_r(r,\sigma) := 1 - \exp\left(\frac{-r^2}{2\cdot\sigma^2}\right) \qquad Q_r(r,\sigma) := \exp\left(\frac{-r^2}{2\cdot\sigma^2}\right)$$



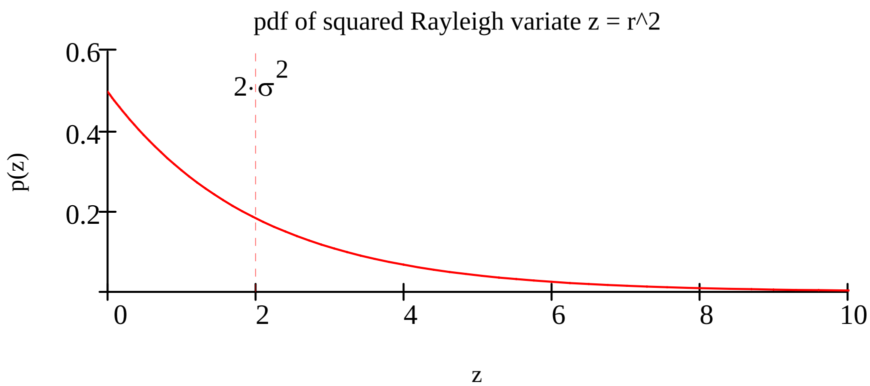
It's often easier to work with the squared amplitude

$$z = r^2 = (|g|)^2$$

because it is exponentially distributed. This follows from a simple change of variables to *z* from *r* in the Rayleigh pdf. The pdf of *z* is

$$p_z(z,\sigma) := \frac{1}{2\cdot\sigma^2}\cdot\exp\left(-\frac{z}{2\cdot\sigma^2}\right) \qquad z \ge 0 \qquad (4.2.5)$$

For σ=1, it looks like this:

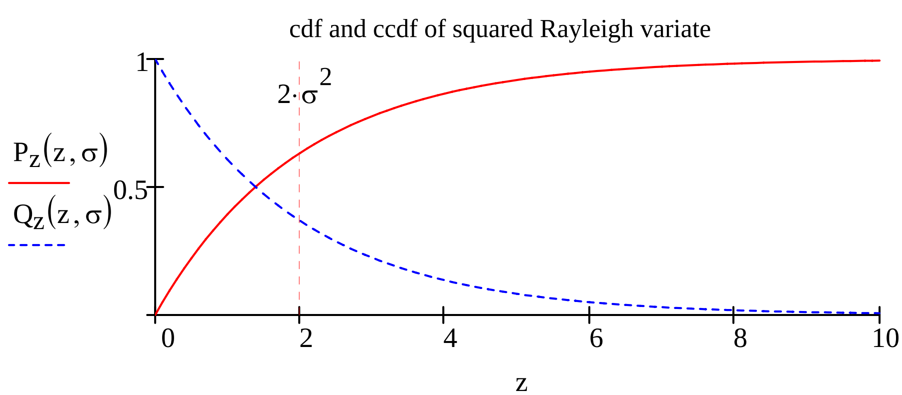


The mean equals the decay constant and so does the standard deviation.

$$\mu_z = 2\cdot\sigma^2 \qquad \sigma_z = 2\cdot\sigma^2$$

The cumulative distribution function and complementary distribution function of *z* are

$$P_z(z,\sigma) := 1 - \exp\left(-\frac{z}{2\cdot\sigma^2}\right) \qquad Q_z(z,\sigma) := \exp\left(\frac{-z}{2\cdot\sigma^2}\right) \qquad (4.2.6)$$



4.2.2 The Chi-Squared Distribution

In signal detection, you often work with vectors of independent, identically distributed (iid) Gaussian noise. A key question then is the distribution of its squared norm; that is, the sum of the squared components. It turns out to be chi-squared.

The chi-squared distribution may be familiar to you from your undergraduate course on sampling statistics. It also plays an important role in wireless communication. Briefly, if the real random variables w_i, $i = 1..n$, are independent and Gaussian with **zero mean** and **variance σ**, then their sum $W = \Sigma_i w_i$ has a chi-squared pdf with n degrees of freedom:

$$p_W(W) = \frac{1}{\sigma^n \cdot 2^{0.5 \cdot n} \cdot \Gamma\left(\frac{n}{2}\right)} \cdot W^{0.5 \cdot (n-2)} \cdot \exp\left(\frac{-W}{2 \cdot \sigma^2}\right) \quad \text{for } W \ge 0 \tag{4.2.13}$$

where $\Gamma(x)$ is the gamma function, equal to $(x\text{-}1)!$ for integer values of x. The first two moments are

$$E(W) = n \cdot \sigma^2 \qquad E\left(W^2\right) = \left(n^2 + 2 \cdot n\right) \cdot \sigma^4 \qquad {\sigma_W}^2 = 2 \cdot n \cdot \sigma^4$$

From our discussion in Section 4.2.1 above, we know that the squared magnitude of a complex random variable $z = |g|^2$ is the sum of two squared and independent real Gaussians in the case of Rayleigh fading. It therefore has a chi-squared pdf with 2 degrees of freedom, which we found to be exponential. In fact, the chi-squared pdf is simpler any time we have an even number of degrees of freedom. For example, the squared norm of a vector of L **complex** Gaussians has a chi-squared pdf with $n = 2L$ degrees of freedom,

$$p_Z(Z) = \frac{1}{\left(2 \cdot \sigma^2\right)^L \cdot (L-1)!} \cdot Z^{L-1} \cdot \exp\left(\frac{-Z}{2 \cdot \sigma^2}\right) \quad \text{for } Z \ge 0 \tag{4.2.14}$$

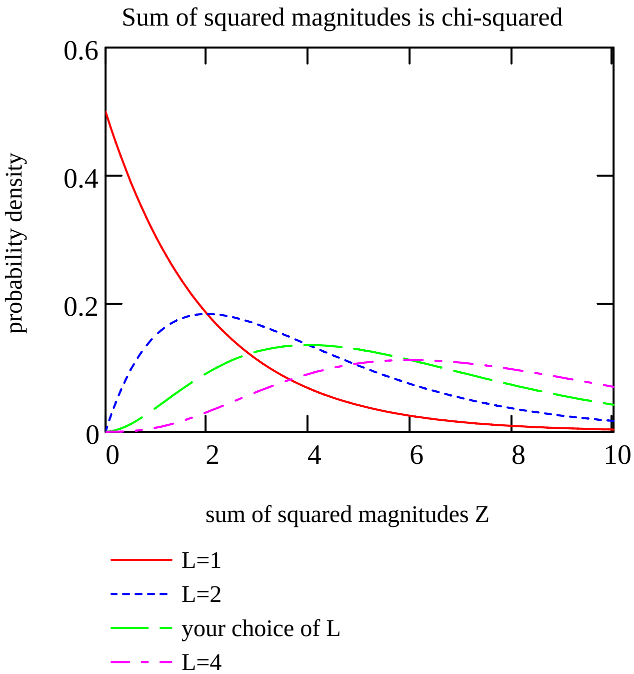
Repeated integration by parts gives the cdf and ccdf

$$P_Z(Z) = 1 - \exp\left(\frac{-Z}{2 \cdot \sigma^2}\right) \cdot \sum_{m=0}^{L-1} \left[\frac{1}{m!} \cdot \left(\frac{Z}{2 \cdot \sigma^2}\right)^m\right] \quad \text{for } Z \ge 0 \tag{4.2.15}$$

$$Q_Z(Z) = \exp\left(\frac{-Z}{2 \cdot \sigma^2}\right) \cdot \sum_{m=0}^{L-1} \left[\frac{1}{m!} \cdot \left(\frac{Z}{2 \cdot \sigma^2}\right)^m\right]$$

The plot below illustrates the chi-squared pdf of the squared norm of a vector of L **complex** noise samples, each with variance σ^2 in their real and imaginary parts (equivalent to the squared norm of a vector of $n = 2L$ real noise samples). You can choose your own value here:

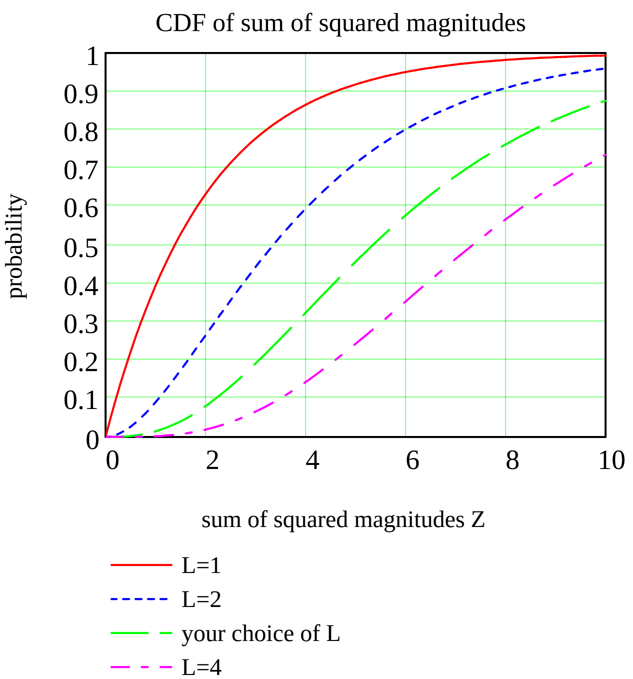
$L := 3$ $\qquad\qquad \sigma^2 = 1$



The mean of the pdf is $2L\sigma^2$. It is the mean of the squared norm of that length-L vector of complex Gaussians.

For larger values of L, it can be approximated by a Gaussian pdf.

and here's the cumulative distribution function



Recall also that each z has an exponential pdf with mean $2\sigma^2$. Thus the chi-squared pdf with two degrees of freedom is exponential. The pdf (4.2.14) of a sum of L such independent variates is therefore the convolution of L exponential pdfs. The Laplace transform (related to the characteristic function and moment generating function) of the chi-squared pdf with an even number of degrees of freedom (4.2.14) is therefore

$$\Phi_Z(s) = \frac{1}{\left(2\cdot\sigma^2\cdot s+1\right)^L} \qquad (4.2.18)$$

4.2.3 Nakagami Density

Just as the chi-squared pdf gives the distribution of the squared norm of a zero-mean Gaussian noise vector, the Nakagami-*m* distribution [**Naka60**] gives the pdf of the norm itself for even values of degrees of freedom. It is therefore a useful generalization of the Rayleigh pdf. Its usual definition is

$$p_{r\_N}\left(r, m, \sigma_N\right) := \frac{2}{\Gamma(m)}\cdot\left(\frac{m}{2\cdot\sigma_N^{\,2}}\right)^m\cdot r^{2\cdot m-1}\cdot\exp\left(\frac{-m\cdot r^2}{2\cdot\sigma_N^{\,2}}\right) \quad \text{for} \quad r \ge 0 \quad \text{and} \quad m \ge \frac{1}{2} \qquad (4.2.23)$$

where m is the order of the pdf and $2\sigma_N^{\,2}$ is the mean square value. For m=1, Nakagami reduces to Rayleigh. The Nakagami distribution accommodates non-integer m values; however, in practice, almost all work with Nakagami distribution is based on integer m.

To relate the Nakagami-*m* distribution so the chi-squared distribution with $n = 2L$ degrees of freedom, make these definitions:

Z the squared norm of the length-L vector of complex Gaussians, as above

$R = \sqrt{Z}$ the norm of the same vector

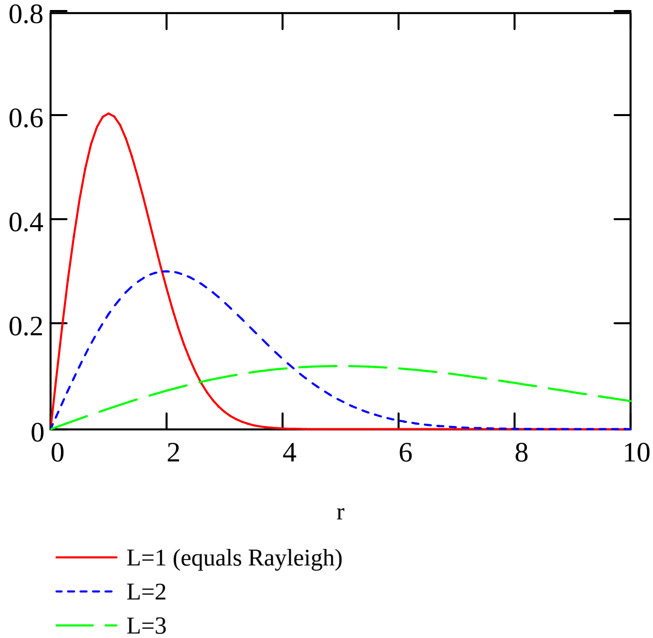
$m = L$ the order of Nakagami and twice the number of chi-squared d.f.

$2\cdot\sigma_N^{\,2} = 2\cdot L\cdot\sigma^2$ where σ^2 is, as always, the variance of the real and imaginary parts of each vector component

The pdf of the norm is then

$$p_R(r, \sigma, L) := \frac{2}{\Gamma(L)}\cdot\left(\frac{1}{2\cdot\sigma^2}\right)^L\cdot r^{2\cdot L-1}\cdot\exp\left(\frac{-r^2}{2\cdot\sigma^2}\right) \qquad (4.2.24)$$

We'll check (4.2.24) below, but first let's have a look at the pdf for $\sigma := 1$:



The "Nakagami-m" pdf

This is "Nakagami-m" in quotation marks, because it has been scaled so that the mean square value is $2L\sigma^2$, not $2\sigma^2$.

Now we confirm the connection between the Nakagami distribution and the the chi-squared distribution. By change of variables in (4.2.23), the squared amplitude $z = r^2$ has a gamma pdf:

$$p_{z\_N}(z,m,\sigma) := \frac{1}{\Gamma(m)}\cdot\left(\frac{m}{2\cdot\sigma^2}\right)^m \cdot z^{m-1}\cdot e^{\left(\frac{-m\cdot z}{2\cdot\sigma^2}\right)} \quad \text{for} \quad z \geq 0 \quad \text{and} \quad m \geq \frac{1}{2}$$

(4.2.25)

For integer m, this is just the chi-squared pdf (4.2.14), if we identify $m = L$ and $2\sigma_N{}^2/m = 2\sigma^2$. Thus the scaled Nakagami-m pdf (4.2.24) is indeed the norm of a length-L vector of complex Gaussians with variance σ^2 in their real and imaginary parts.

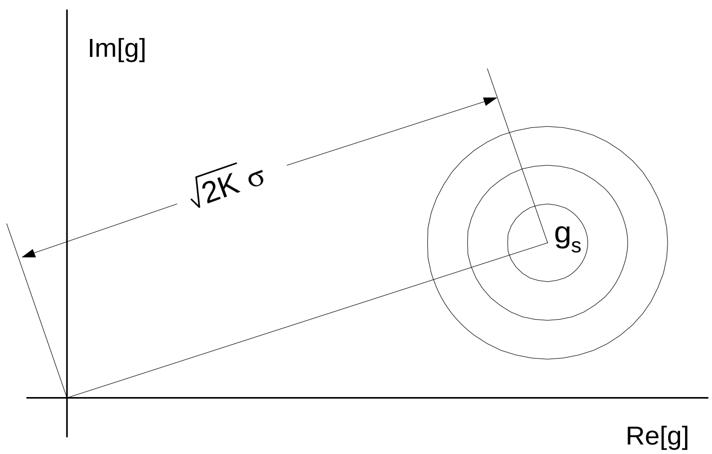
Distributions Derived From *NON*-Zero Mean Gaussian Variates

4.2.4 The Rice pdf

In mobile satellite systems, or in land mobile radio in suburban and rural areas, the signal is often received with a LOS component which produces Rice fading. The total gain

$$g = g_s + g_d \tag{4.2.7}$$

is the sum of a constant specular (or LOS or discrete) component g_s and a zero mean Gaussian diffuse (or scattered) component g_d, so that g is a nonzero mean Gaussian variate. The specular component has K times the power of the diffuse component (the Rice K-factor), so that K=0 gives Rayleigh fading and K==>∞ gives a constant channel. But be careful - some literature (mostly in the mobile satellite area) uses K as the ratio of diffuse to specular power, the reciprocal of the conventional definition. The sketch shows the isoprobability contours.



Denote the variance of the diffuse component by σ^2. From the power ratio K we have the magnitude of the specular component.

$$|g_s| = \sqrt{2 \cdot K} \cdot \sigma \qquad \text{since} \qquad \frac{1}{2} \cdot E\left[\left(|g_d|\right)^2\right] = \sigma^2 \tag{4.2.8}$$

The total average power in g is then

$$\frac{1}{2} \cdot E\left[\left(|g|\right)^2\right] = \frac{1}{2} \cdot E\left[\left(|g_s|\right)^2\right] + \frac{1}{2} \cdot E\left[\left(|g_d|\right)^2\right] = K \cdot \sigma^2 + \sigma^2 = \sigma^2 \cdot (1 + K) \tag{4.2.9}$$

and the mean and variance of g are $\mu_g = g_s$ and $\sigma_g^{\,2} = \sigma^2$. Its pdf is Gaussian:

$$p_g(g) = \frac{1}{2 \cdot \pi \cdot \sigma^2} \cdot \exp\left[-\frac{1}{2} \cdot \frac{\left(|g - g_s|\right)^2}{\sigma^2}\right] \tag{4.2.10}$$

Changing to polar coordinates makes $z = r^2$ non-central χ^2 (chi-squared) with mean $\sigma^2(1+K)$ and two degrees of freedom, as discussed below. Alternatively, the pdf of r is Rician:

$$p_{r\_Rice}(r,K,\sigma) := \frac{r}{\sigma^2}\cdot\exp\left(-\frac{r^2}{2\cdot\sigma^2} - K\right)\cdot I0\left(\frac{r\cdot\sqrt{2\cdot K}}{\sigma}\right) \qquad (4.2.11)$$

From the isoprobability sketch above, it is clear that the phase angle is not independent of the amplitude. The unconditional pdf of the phase angle for a real specular component (i.e., zero mean phase angle) is obtained by adapting [**Proa89**, eqn. 4.2.103]. First, the Q function:

$$Q(x) = \frac{1}{\sqrt{2\cdot\pi}}\cdot\int_x^{\infty} e^{-\frac{\alpha^2}{2}}\,d\alpha$$

or $Q(x) := \mathrm{cnorm}(-x)$

$$p_{\theta\_Rice}(\theta,K,\sigma) := \frac{1}{2\cdot\pi}\cdot e^{-K}\cdot\left[1+\sqrt{4\cdot\pi\cdot K}\cdot\cos(\theta)\cdot e^{K\cdot\cos(\theta)^2}\cdot\left(1-Q\left(\sqrt{2\cdot K}\cdot\cos(\theta)\right)\right)\right]$$

(4.2.12)

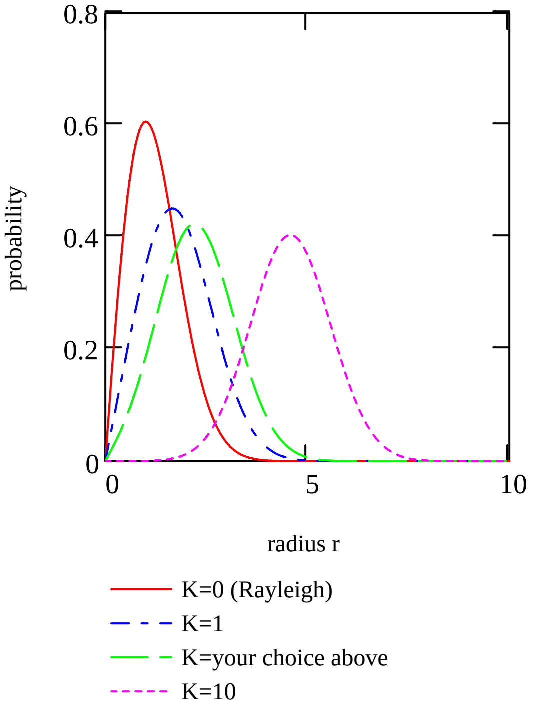
Now let's see what these pdfs look like.

plot ranges: $r := 0, 0.05 .. 10$ $\qquad \theta := -\pi, -0.98\cdot\pi .. \pi$

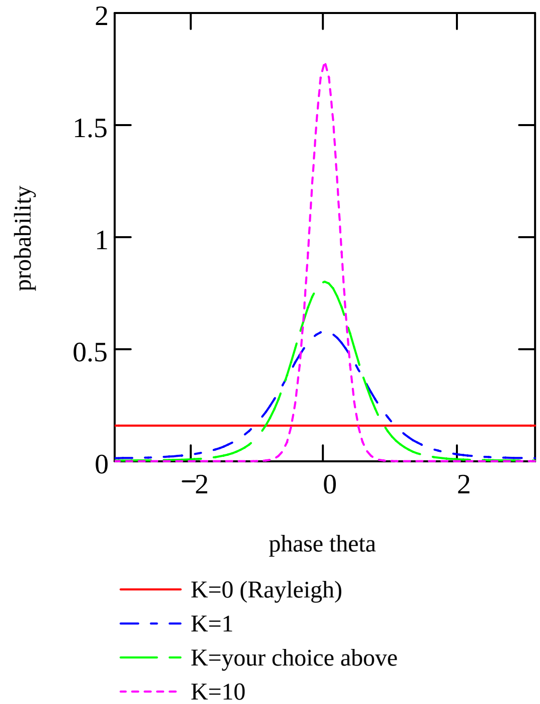
Your choice of K and σ below

$K := 2$ $\qquad \sigma := 1$

The plots show that the probability of deep fades is greatly reduced in Rice fading, as is the probability of a large random phase shift.



Rice amplitude pdf



Rice phase pdf (for zero mean)

Inspection of the graphs suggests that they can be approximated by Gaussian pdfs for large *K*. It's easy to see why this is so if you rotate the coordinates for g_d to resolve it into a radial component (along the same line as g_s) and a transverse component (orthogonal to the radial component). For large *K*, the transverse component makes little difference to the amplitude, which is then well modeled by the Gaussian radial component with the specular component as a mean. Similarly, the radial component makes little difference to the phase, which is then well modeled by the Gaussian transverse component divided by the specular amplitude. Therefore,

- approx amplitude pdf, large *K*: Gaussian, mean $\sqrt{2 \cdot K} \cdot \sigma$ and standard deviation σ
- approx phase pdf, large *K*: Gaussian, mean $\arg(g_s)$ and standard deviation $\dfrac{1}{\sqrt{2 \cdot K}}$

4.2.5 The Non-Central Chi-Squared Distribution

There are counterpart results in the case of Rice fading, where the centre of the Gaussian distribution of *g* is away from the origin. We have already seen that the amplitude gain *r* has a Rice pdf. The power gain $z = r^2 = |g|^2$ has a non-central chi-squared distribution with two degrees of freedom, given by

$$p_Z(Z) = \frac{1}{2\cdot\sigma^2}\cdot\exp\left[-\left(\frac{Z}{2\cdot\sigma^2}+K\right)\right]\cdot I_0\left(\sqrt{\frac{2\cdot K\cdot Z}{\sigma^2}}\right) \tag{4.2.19}$$

where $I_0(x)$ is the modified Bessel function of the first kind with order 0. The pdf of a sum $Z = \Sigma_i z_i$ of L such independent power gains, as obtained in diversity systems, has a non-central chi-squared distribution with $2L$ degrees of freedom. First define the sum of the K-factors

$$K = \sum_{m=1}^{L} K_i \tag{4.2.20}$$

Then the pdf of Z is obtained by a slight adaptation of [**Proa95**] as

$$p_Z(Z) = \frac{1}{2\cdot\sigma^2}\cdot\left(\frac{Z}{2\cdot\sigma^2\cdot K}\right)^{0.5\cdot(L-1)}\cdot\exp\left[-\left(\frac{Z}{2\cdot\sigma^2}+K\right)\right]\cdot \mathrm{In}\left(L-1,\sqrt{\frac{2\cdot K\cdot Z}{\sigma^2}}\right)$$

$$= \left(\frac{Z}{K}\right)^{0.5\cdot(L-1)}\cdot e^{-(Z+K)}\cdot \mathrm{In}\left(L-1, 2\cdot\sqrt{K\cdot Z}\right) \qquad \text{for } \sigma^2 = 1/2 \tag{4.2.21}$$

where $In(N,x)$ is Mathcad notation for the modified Bessel function of the first kind with order N. Its characteristic function is adapted from [**Schw66**, App. B] as

$$\Phi_Z(s) = \left(2\cdot\sigma^2\cdot s+1\right)^{-L}\cdot\exp\left(\frac{-2\cdot\sigma^2\cdot s\cdot K}{1+2\cdot\sigma^2\cdot s}\right)$$

$$= \frac{\exp\left(\frac{-s\cdot K}{s+1}\right)}{(s+1)^L} \qquad \text{for } \sigma^2 = 1/2 \tag{4.2.22}$$