2. SIGNAL REPRESENTATION

- Before we can think clearly about detecting signals, we must be able to describe them.

- This section is partly review, partly extension of what you have learned in your undergrad course on digital communications. Topics:
  
  - complex lowpass representation of bandpass signals, both deterministic and random
  
  - signal spaces – a link between time functions and vector diagrams, with generalization to other useful inner product spaces
  
  - statistics of random processes projected onto basis sets.
2.1 Complex Baseband Representation of Bandpass Signals

- Although we have studied detection of lowpass signals, it is more common to transmit bandpass signals, ones that result from modulating a carrier. Why?
  - Move the signals away from DC and centre them in the sweet spot of the transmission characteristic (e.g., telephone line modems).
  - A need to use radio or ultrasound, etc, for the link.
  - More bandwidth is available at higher frequencies; e.g., 1% bandwidth ($W/f_c$) is: 10 kHz at 1 MHz; 10 MHz at 1 GHz; 280 MHz at 28 GHz.

- We will use a slightly unconventional notation: “complex baseband” instead of “complex envelope.” Why?
  - It makes the energy of a baseband signal equal that of its bandpass counterpart (no more factor of $\frac{1}{2}$).
  - The real and imaginary components each have the same signal model as in classical detection of real signals.
  - Variances and autocorrelations have the same definitions as in signal processing literature.
2.1.1 Bandpass Signals

- Here’s a basic model of modulation:

![Modulation Diagram](image)

- The bandwidth at passband is twice as large as at baseband. Why be content with just one signal in that bandwidth? Pack in a second signal, this one on a sine carrier. “Quadrature multiplexing.”

![Quad Multiplexing Diagram](image)
The quadrature multiplex is good, provided we can separate the signals at the receiver. Can we?

Ignore the channel filter and the noise, so \( C(f) = 1 \), \( N_0 = 0 \). Then

\[
\tilde{r}(t) = \tilde{v}(t) = \sqrt{2}v_r(t)\cos(2\pi f_c t) - \sqrt{2}v_i(t)\sin(2\pi f_c t)
\]

The receiver’s cosine branch forms

\[
\sqrt{2}\cos(2\pi f_c t)\tilde{r}(t) = 2v_r(t)\cos^2(2\pi f_c t) - 2v_i(t)\cos(2\pi f_c t)\sin(2\pi f_c t)
\]

and lowpasses it to recover

\[
r_r(t) = v_r(t) \quad \text{Similarly, } r_i(t) = v_i(t).
\]

A clean separation, since cos and sin are orthogonal.
2.1.2 The Complex Baseband Signal

- In complex terms, modulation looks like this:

  \[ v(t) = v_r(t) + jv_i(t) \]

  The complex signal \( v(t) \) is the “complex baseband” counterpart of the bandpass signal \( \tilde{v}(t) \) produced by the quadrature modulator.

- Since \( \tilde{v}(t) = \sqrt{2} \text{Re} \left[ v(t)e^{j2\pi f_ct} \right] \), the complex baseband signal \( v(t) \) is a “time varying phasor” (an rms phasor).
In the frequency domain, modulation looks like this:

\[\begin{align*}
\text{Re}(w(t)) \quad &\quad \text{Im}(w(t)) \\
\text{Re}(w(t) + w^*(-f)) \quad &\quad \text{Im}(w(t) - w^*(f)) \\
\text{Re}(w(t) + w^*(-f)) \quad &\quad \text{Im}(w(t) - w^*(f))
\end{align*}\]

Recall that if \( w(t) \) has transform \( W(f) \), then \( \text{Re}[w(t)] \) has transform \( \frac{1}{2} \left( W(f) + W^*(-f) \right) \). Apply this to \( \sqrt{2} v(t) \exp(j2\pi f_ct) \).

Since \( v(t) \) is complex, its transform need not be conjugate symmetric. See Fourier symmetry relations for complex signals in Appendix A. If \( V(f) \neq V^*(-f) \), is \( v(t) \) necessarily complex?

Practical issues:

- If “cos” and “sin” from LO are not separated by exactly 90°, or if gains in branches are not exactly equal, then imbalance error. Some problems: FM trajectory not exactly circular; SSB generation with Hilbert transform has imperfect sideband cancellation; distorted constellations.
- The \( v_r(t) \) and \( v_i(t) \) inputs are usually from DSP through D/As and reconstruction filters.
➢ If DC offset in DACs or filters, or if carrier feedthrough in mixers, then offset error:

➢ Both problems can be eliminated with a DSP quadrature modulator (Appendix B).

• Quadrature demodulation recovers the complex baseband equivalent of the received signal in the reverse fashion:
• Mathematical description. First, a useful identity:

\[
\frac{1}{2} \text{Re}[\alpha \beta^* + \alpha \beta]
\]

Next, recovery of the real component \( r_r(t) \) is by

\[
\sqrt{2} \text{Re} \left[ r(t) e^{j2\pi f_c t} \right] \sqrt{2} \cos(2\pi f_c t)
\]

\[
= 2 \text{Re} \left[ r(t) e^{j2\pi f_c t} - \text{Re} \left[ e^{j2\pi f_c t} \right] \right]
\]

\[
= \text{Re} \left[ r(t) + \frac{r(t)}{2} e^{j4\pi f_c t} \right] \xrightarrow{\text{lowpass}} r_r(t)
\]

Similarly for \( r_i(t) \), using \(-\sqrt{2} \sin(2\pi f_c t) = \sqrt{2} \text{Re} \left[ j e^{j2\pi f_c t} \right] \).

• Practical issues:

➢ The phase of the recovered \( r(t) \) is defined with respect to the receiver’s LO.

➢ Phase or gain imbalance will distort recovered \( r(t) \).

➢ Carrier feedthrough and self-mixing causes a DC offset error in \( r(t) \).

➢ Imbalance and feedthrough both eliminated by DSP quadrature demodulator (Appendix B).

• Finally, note that another common representation of bandpass signals is the “complex envelope,” equal to \( \sqrt{2} \) times the complex baseband signal. It
2.1-8

complicates the energy and power relationships. Think of complex baseband as the rms equivalent of the complex envelope.

2.1.3 Generality of Complex Baseband

• Any real, bandpass signal can be written in complex baseband form. Consider this one:

The “carrier frequency” $f_c$ is not necessarily the centre.

• Because $\tilde{v}(t)$ is real, $\tilde{V}(f) = \tilde{V}^*(-f)$ (conjugate symmetric). So

$$
\tilde{v}(t) = \int_{-\infty}^{\infty} \tilde{V}(f) e^{j2\pi ft} df = 2 \text{Re} \left[ \int_{f_c-W_1}^{f_c+W_2} \tilde{V}(f) e^{j2\pi ft} df \right]
$$

$$
= \sqrt{2} \text{Re} \left[ e^{j2\pi f_c t} \int_{-W_1}^{W_2} \sqrt{2} \tilde{V}(\alpha + f_c) e^{j2\pi \alpha t} d\alpha \right] = \sqrt{2} \text{Re} \left[ v(t) e^{j2\pi f_c t} \right]
$$

lowpass, complex baseband
• So the Fourier transform of \( v(t) \) is \( \sqrt{2} \) times the frequency translate

\[
V(f) = \sqrt{2} \tilde{V}_+ (f + f_c)
\]

(subscript + means positive frequencies only)

Equivalently, each image in \( \tilde{V}(f) \) is \( \frac{1}{\sqrt{2}} \) times as large as the baseband \( V(f) \).

2.1.4 Bandpass Filters

• In diagrams, we often omit the quadrature modulator and demodulator, and just show the envelope, even for filters.
2.1-10

- Clearly, the bandpass counterparts satisfy $\tilde{Y}(f) = \tilde{C}(f)\tilde{V}(f)$:

So $Y(f) = C(f)V(f)$

$V(f) = \sqrt{2}\tilde{V}_+(f + f_c)$

$C(f) = \tilde{C}_+(f + f_c)$ (note no $\sqrt{2}$)

$Y(f) = \sqrt{2}\tilde{Y}_+(f + f_c)$

- Time domain version:

$$\tilde{y}(t) = \tilde{v}(t) \otimes \tilde{c}(t) = \int_{-\infty}^{\infty} \tilde{c}(\alpha) \tilde{v}(t - \alpha) \, d\alpha$$

$$= \int_{-\infty}^{\infty} \tilde{c}(\alpha) \sqrt{2} \text{Re} \left[ v(t - \alpha) e^{j2\pi f_c(t-\alpha)} \right] \, d\alpha$$

$$= \sqrt{2} \text{Re} \left[ e^{j2\pi f_c t} \int_{-\infty}^{\infty} \tilde{c}(\alpha) e^{-j2\pi f_c \alpha} v(t - \alpha) \, d\alpha \right]$$
So \( y(t) = c(t) \otimes v(t) \), where \( c(t) \) is defined without the \( \sqrt{2} \):

\[
c(t) = \tilde{c}(t) e^{-j2\pi f_c t} \bigg|_{\text{lowpass}} = c_+(t) e^{-j2\pi f_c t} \quad \text{left shifted}
\]

### 2.1.4 Phase and Frequency Errors

- Back to the complex link model:

  ![Complex Link Model Diagram]

  - What if the channel filter is flat in frequency, but causes a phase shift \( \theta \)?

    \[
    C(f) = e^{j\theta} \quad \text{or} \quad c(t) = e^{j\theta} \delta(t)
    \]

    Then we recover the complex envelope (ignoring noise) \( r(t) = e^{j\theta} v(t) \):

    \[
    r_r(t) = \cos(\theta) v_r(t) - \sin(\theta) v_i(t)
    \]
    \[
    r_i(t) = \sin(\theta) v_r(t) + \cos(\theta) v_i(t)
    \]

    Oops – “crosstalk” between the two channels.

- We have the same problem of signal rotation if the transmit oscillator has a phase shift of \( \theta \) with respect to the receive oscillator (or if Rx lags Tx by \( \theta \) - it’s symmetric).
Since the transmit and receive oscillators are not linked, there is normally a phase shift between them. Again the complex baseband signal is rotated by \( \theta \).

- Similarly, a frequency offset is a linearly time-varying phase shift. The Rx might view the Tx LO as producing \( e^{j2\pi(f_c+f_o)t + \theta} \), so if the nominal Tx signal is \( s(t) \), the recovered signal is \( s(t)e^{j2\pi f_o t} e^{j\theta} \).