2. SIGNAL REPRESENTATION

- Before we can think clearly about detecting signals, we must be able to describe them.

- This section is partly review, partly extension of what you have learned in your undergrad course on digital communications. Topics:

  - complex lowpass representation of bandpass signals, both deterministic and random

  - signal spaces – a link between time functions and vector diagrams, with generalization to other useful inner product spaces

  - statistics of random processes projected onto basis sets.

2.1 Complex Baseband Representation of Bandpass Signals

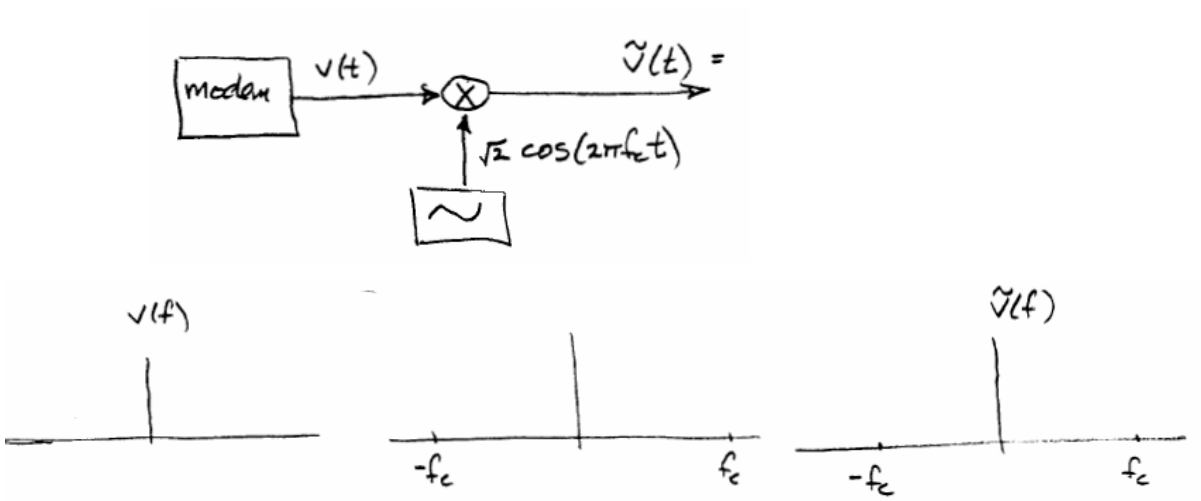
- Although we have studied detection of lowpass signals, it is more common to transmit bandpass signals, ones that result from modulating a carrier. Why?
  - Move the signals away from DC and centre them in the sweet spot of the transmission characteristic (e.g., telephone line modems).
  - A need to use radio or ultrasound, etc, for the link.
  - More bandwidth is available at higher frequencies; e.g., 1% bandwidth (W/f_c) is: 10 kHz at 1 MHz; 10 MHz at 1 GHz; 280 MHz at 28 GHz.

    narrowband wideband

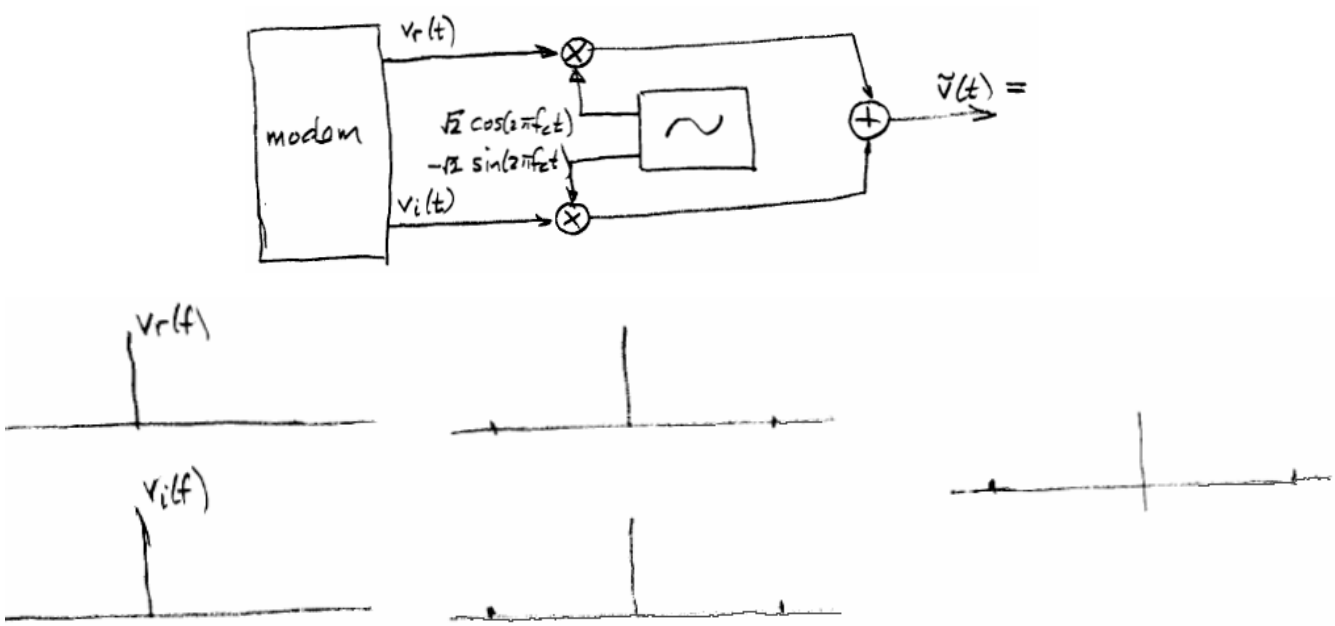
- We will use a slightly unconventional notation: "complex baseband" instead of "complex envelope." Why?
  - It makes the energy of a baseband signal equal that of its bandpass counterpart (no more factor of ½).
  - The real and imaginary components each have the same signal model as in classical detection of real signals.
  - Variances and autocorrelations have the same definitions as in signal processing literature.

2.1.1 Bandpass Signals

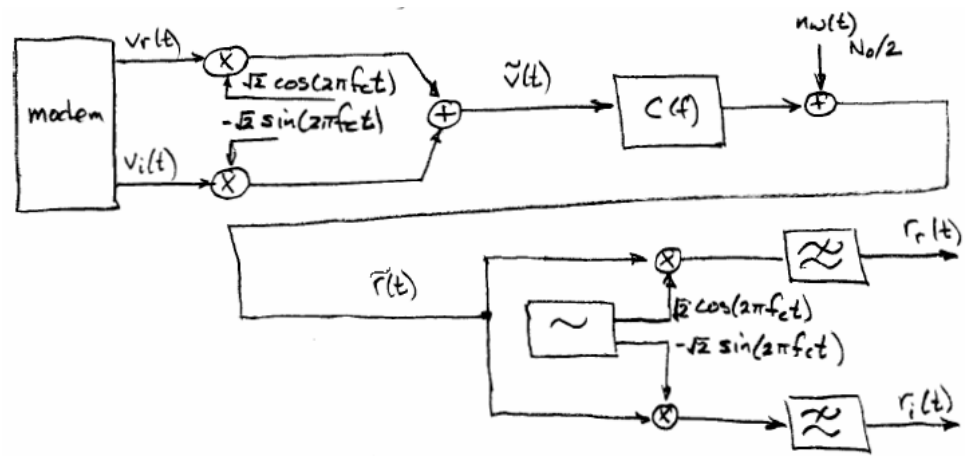
- Here's a basic model of modulation:



- The bandwidth at passband is twice as large as at baseband. Why be content with just one signal in that bandwidth? Pack in a second signal, this one on a sine carrier. "Quadrature multiplexing."



- The quadrature multiplex is good, *provided* we can separate the signals at the receiver. Can we?



Ignore the channel filter and the noise, so $C(f) = 1$, $N_0 = 0$. Then

$$\tilde{r}(t) = \tilde{v}(t) = \sqrt{2}v_r(t)\cos\left(2\pi f_c t\right) - \sqrt{2}v_i(t)\sin\left(2\pi f_c t\right)$$

The receiver's cosine branch forms

$$\sqrt{2}\cos\left(2\pi f_c t\right)\tilde{r}(t) = 2v_r(t)\cos^2\left(2\pi f_c t\right) - 2v_i(t)\cos\left(2\pi f_c t\right)\sin\left(2\pi f_c t\right)$$

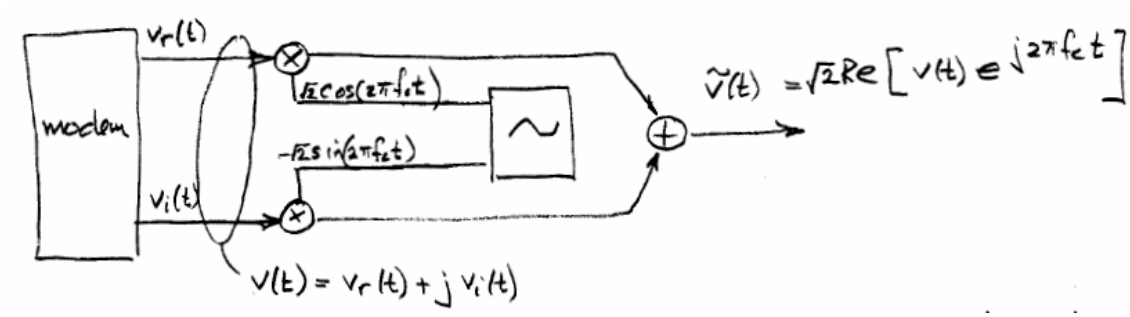
and lowpasses it to recover

$$r_r(t) = v_r(t) \qquad \text{Similarly, } r_i(t) = v_i(t).$$

A clean separation, since cos and sin are orthogonal.

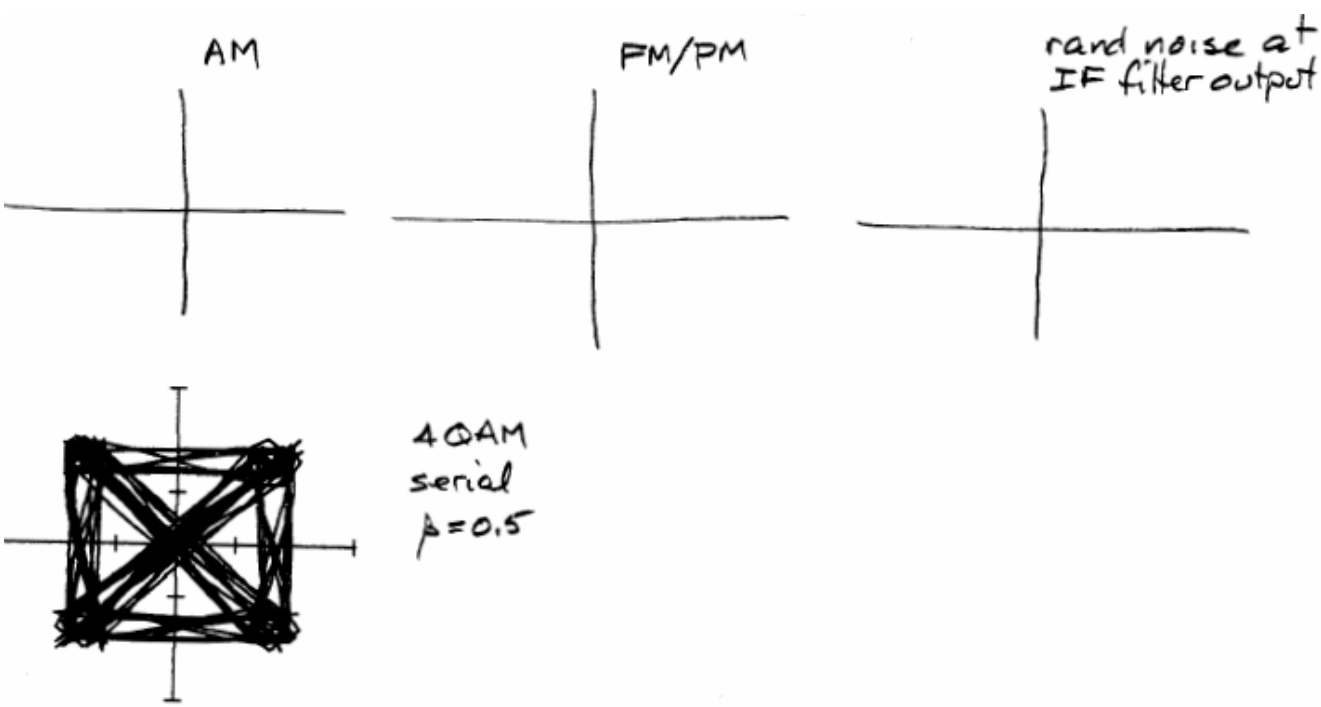
2.1.2 The Complex Baseband Signal

- In complex terms, modulation looks like this:

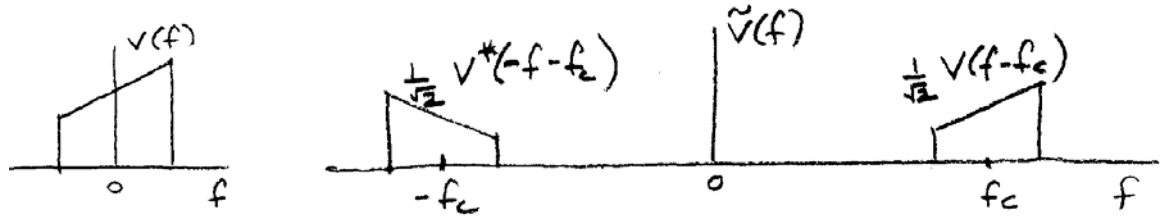


The complex signal $v(t)$ is the "complex baseband" counterpart of the bandpass signal $\tilde{v}(t)$ produced by the quadrature modulator.

- Since $\tilde{v}(t) = \sqrt{2}\,\mathrm{Re}\left[v(t)e^{j2\pi f_c t}\right]$, the complex baseband signal $v(t)$ is a "time varying phasor" (an rms phasor).



- In the frequency domain, modulation looks like this:

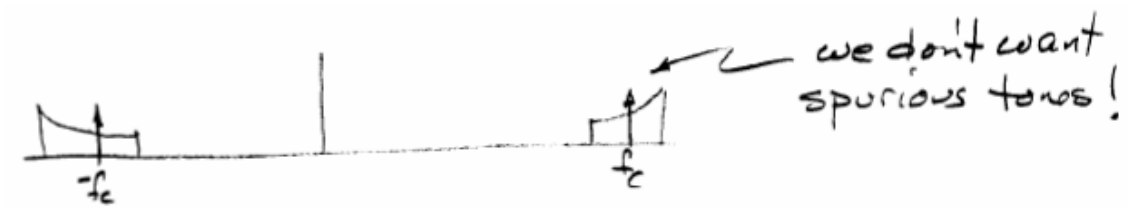


Recall that if $w(t)$ has transform $W(f)$, then $\mathrm{Re}\left[w(t)\right]$ has transform $\frac{1}{2}\left(W(f)+W^*(-f)\right)$. Apply this to $\sqrt{2}\,v(t)\exp\left(j2\pi f_c t\right)$.

- Since $v(t)$ is complex, its transform need not be conjugate symmetric. See Fourier symmetry relations for complex signals in Appendix A. If $V(f)\neq V^*(-f)$, is $v(t)$ necessarily complex?

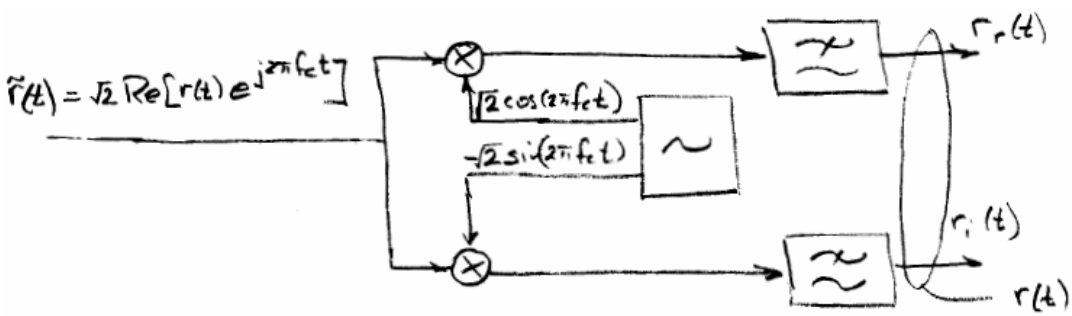
- Practical issues:
  - If "cos" and "sin" from LO are not separated by exactly 90°, or if gains in branches are not exactly equal, then imbalance error. Some problems: FM trajectory not exactly circular; SSB generation with Hilbert transform has imperfect sideband cancellation; distorted constellations.
  - The $v_r(t)$ and $v_i(t)$ inputs are usually from DSP through D/As and reconstruction filters.

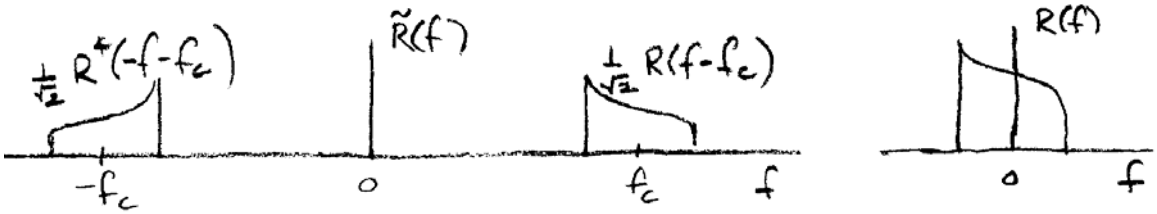
- If DC offset in DACs or filters, or if carrier feedthrough in mixers, then offset error:



- Both problems can be eliminated with a DSP quadrature modulator (Appendix B).

- Quadrature demodulation recovers the complex baseband equivalent of the received signal in the reverse fashion:





- Mathematical description. First, a useful identity:

$$\mathrm{Re}[\alpha]\mathrm{Re}[\beta] = \tfrac{1}{2}\mathrm{Re}\left[\alpha\beta^* + \alpha\beta\right]$$

Next, recovery of the real component $r_r(t)$ is by

$$\begin{aligned}
&\sqrt{2}\,\mathrm{Re}\left[r(t)e^{j2\pi f_c t}\right]\sqrt{2}\cos(2\pi f_c t) \\
&= 2\,\mathrm{Re}\left[r(t)e^{j2\pi f_c t}\right]\mathrm{Re}\left[e^{j2\pi f_c t}\right] \\
&= \mathrm{Re}\left[r(t) + \underbrace{r(t)e^{j4\pi f_c t}}_{\text{double freq term}}\right] \xrightarrow{\text{lowpass}} r_r(t)
\end{aligned}$$

Similarly for $r_i(t)$, using $-\sqrt{2}\,\sin(2\pi f_c t) = \sqrt{2}\,\mathrm{Re}\left[j\,e^{j2\pi f_c t}\right]$.

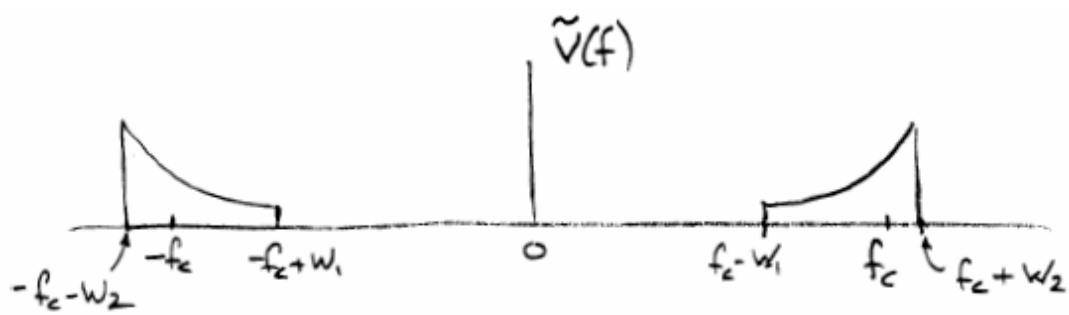
- Practical issues:
  - The phase of the recovered $r(t)$ is defined with respect to the *receiver's* LO.
  - Phase or gain imbalance will distort recovered $r(t)$.
  - Carrier feedthrough and self-mixing causes a DC offset error in $r(t)$.
  - Imbalance and feedthrough both eliminated by DSP quadrature demodulator (Appendix B).

- Finally, note that another common representation of bandpass signals is the "complex envelope," equal to $\sqrt{2}$ times the complex baseband signal. It

complicates the energy and power relationships. Think of complex baseband as the rms equivalent of the complex envelope.

2.1.3 Generality of Complex Baseband

- Any real, bandpass signal can be written in complex baseband form. Consider this one:



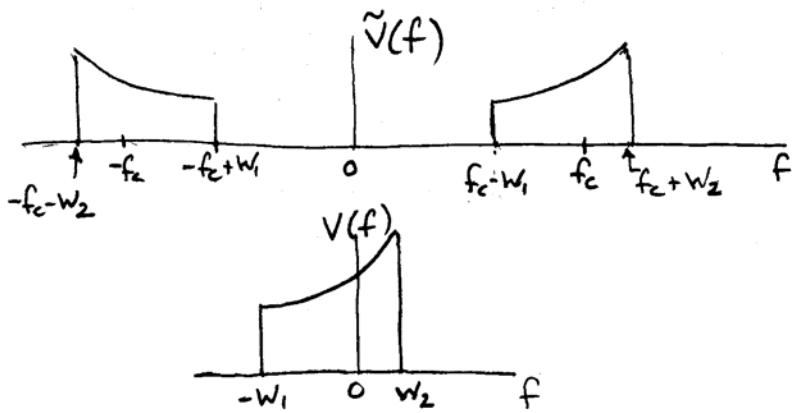
The "carrier frequency" f_c is not necessarily the centre.

- Because $\tilde{v}(t)$ is real, $\tilde{V}(f) = \tilde{V}^*(-f)$ (conjugate symmetric). So

$$\tilde{v}(t) = \int_{-\infty}^{\infty} \tilde{V}(f) e^{j2\pi ft} df = 2\,\mathrm{Re}\left[\int_{f_c - W_1}^{f_c + W_2} \tilde{V}(f) e^{j2\pi ft} df \right]$$

$$= \sqrt{2}\,\mathrm{Re}\left[e^{j2\pi f_c t} \underbrace{\int_{-W_1}^{W_2} \sqrt{2}\,\tilde{V}(\alpha + f_c) e^{j2\pi\alpha t} d\alpha}_{\text{lowpass, complex baseband}} \right] = \sqrt{2}\,\mathrm{Re}\left[v(t) e^{j2\pi f_c t} \right]$$

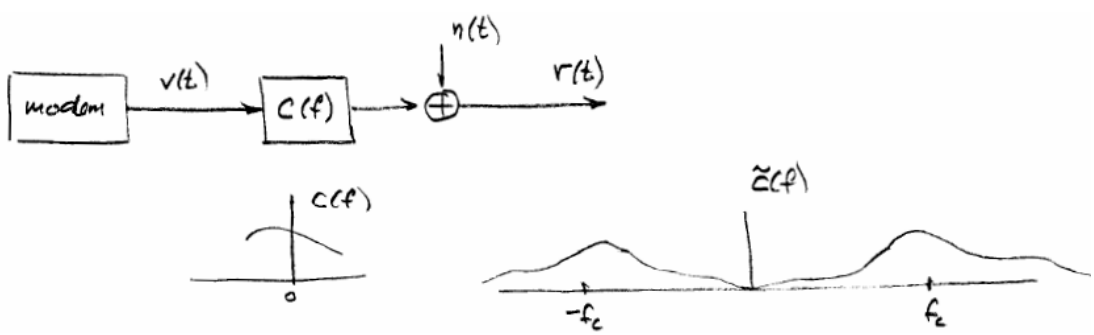
- So the Fourier transform of $v(t)$ is $\sqrt{2}$ times the frequency translate $V(f) = \sqrt{2}\tilde{V}_+(f + f_c)$ (subscript $_+$ means positive frequencies only)



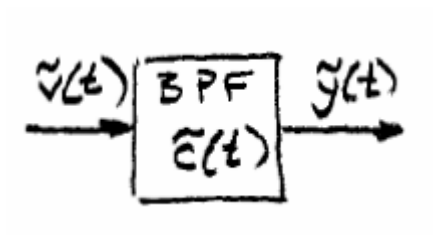
Equivalently, each image in $\tilde{V}(f)$ is $\frac{1}{\sqrt{2}}$ times as large as the baseband $V(f)$.

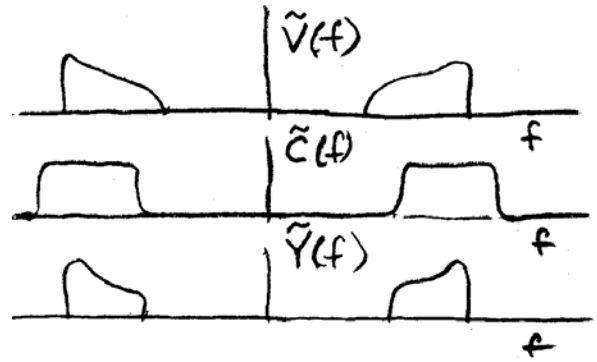
2.1.4 Bandpass Filters

- In diagrams, we often omit the quadrature modulator and demodulator, and just show the envelope, even for filters.

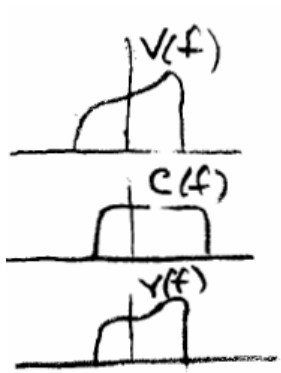


- Clearly, the bandpass counterparts satisfy $\tilde{Y}(f) = \tilde{C}(f)\tilde{V}(f)$:





So $Y(f) = C(f)V(f)$



$$V(f) = \sqrt{2}\,\tilde{V}_+(f + f_c)$$

$$C(f) = \tilde{C}_+(f + f_c) \text{ (note no } \sqrt{2}\text{)}$$

$$Y(f) = \sqrt{2}\,\tilde{Y}_+(f + f_c)$$

- Time domain version:

$$\tilde{y}(t) = \tilde{v}(t) \otimes \tilde{c}(t) = \int_{-\infty}^{\infty} \tilde{c}(\alpha)\tilde{v}(t-\alpha)\,d\alpha$$

$$= \int_{-\infty}^{\infty} \tilde{c}(\alpha)\sqrt{2}\,\mathrm{Re}\left[v(t-\alpha)e^{j2\pi f_c(t-\alpha)}\right]d\alpha$$

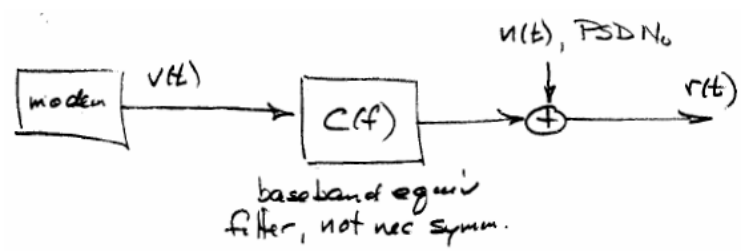
$$= \sqrt{2}\,\mathrm{Re}\left[\underbrace{e^{j2\pi f_c t}\int_{-\infty}^{\infty} \underbrace{\tilde{c}(\alpha)e^{-j2\pi f_c \alpha}}_{\triangleq c(\alpha)} v(t-\alpha)\,d\alpha}_{y(t)}\right]$$

So $y(t) = c(t) \otimes v(t)$, where $c(t)$ is defined *without* the $\sqrt{2}$:

$$c(t) = \tilde{c}(t) e^{-j2\pi f_c t}\Big|_{\substack{\text{lowpass} \\ \text{since z(t) is} \\ \text{lowpass}}} = c_+(t) e^{-j2\pi f_c t} \quad \text{left shifted}$$

2.1.4 Phase and Frequency Errors

- Back to the complex link model:



  - What if the channel filter is flat in frequency, but causes a phase shift θ?

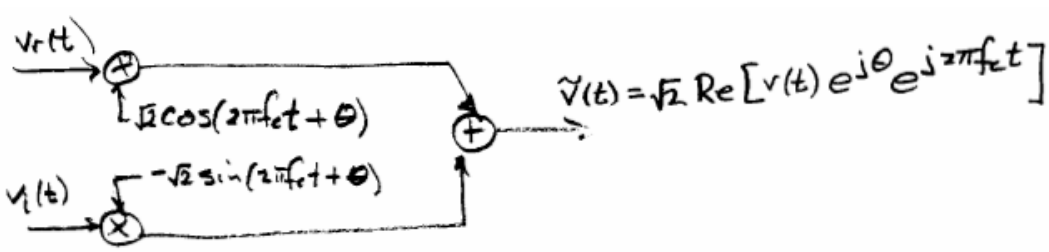
$$C(f) = e^{j\theta} \qquad \text{or} \qquad c(t) = e^{j\theta}\,\delta(t)$$

Then we recover the complex envelope (ignoring noise) $r(t) = e^{j\theta} v(t)$:

$$r_r(t) = \cos(\theta)\, v_r(t) - \sin(\theta)\, v_i(t)$$
$$r_i(t) = \sin(\theta)\, v_r(t) + \cos(\theta)\, v_i(t)$$

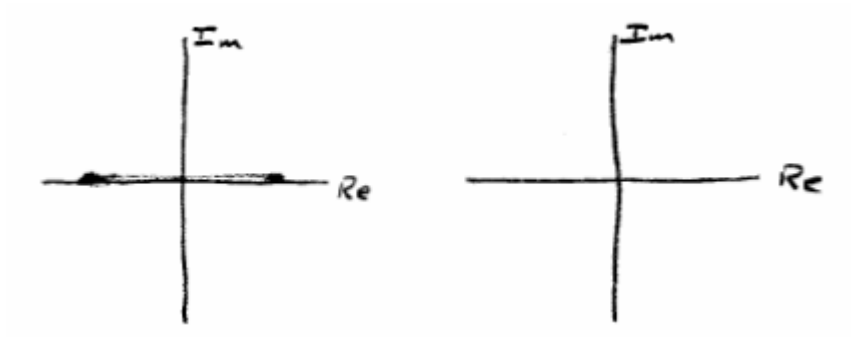
Oops – "crosstalk" between the two channels.

- We have the same problem of signal rotation if the transmit oscillator has a phase shift of θ with respect to the receive oscillator (or if Rx lags Tx by θ - it's symmetric).



Since the transmit and receive oscillators are not linked, there is normally a phase shift between them. Again the complex baseband signal is rotated by θ.

nominal BPSK signal after phase rotation



- Similarly, a frequency offset is a linearly time-varying phase shift. The Rx might view the Tx LO as producing $e^{j2\pi(f_c+f_o)t+\theta}$, so if the nominal Tx signal is $s(t)$, the recovered signal is $s(t)e^{j2\pi f_o t}e^{j\theta}$.

nominal BPSK signal after frequency offset