# CHARACTERIZATION OF A SIMPLE COMMUNICATION NETWORK USING LEGENDRE TRANSFORM

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## ROAD MAP

- We describe an application of the Legendre transform to communication networks.
- Extension of the Legendre transform to non-concave/ non-convex functions.
- Legendre transform was employed to analyze a simple communication network.
- We propose an identification method for its transfer characteristic
- Results are confirmed using the *ns-2* network simulator.

## INTRODUCTION

- Majority of communication network systems are nonlinear.
- Their analysis is rather complex because of this inherent nonlinearity.
- Discrete event systems can be described using linear max-plus or min-plus equations, even though they are nonlinear.
- Communication networks have been analyzed using max-plus algebra and the min-plus algebra:
  - TCP window flow control is max-plus linear [*Baccelli*, 2000].
  - fractal scaling of TCP traffic was observed [*Baccelli*, 2002].
  - network calculus was used for window flow control, multimedia smoothing, and establishing bounds for packet loss rates [*Le Boudec* and *Thiran*, 2002].

## MOTIVATION

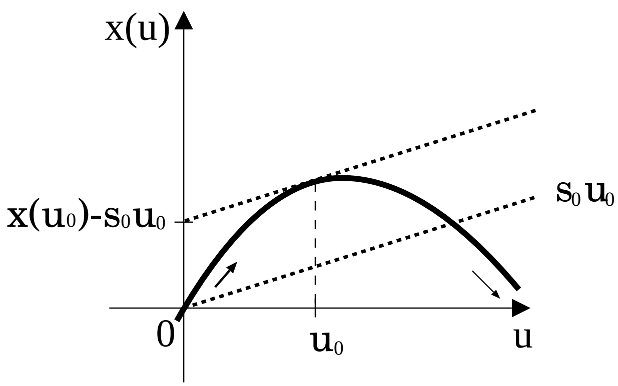
- Legendre transform in max-plus algebra linear systems corresponds to the Fourier transform in conventional linear system theory.
- It is usually applied only to convex or concave functions [*Baccelli*, 1992].
- Our approach employs the extended Legendre transform that can be applied to non-convex/non-concave functions.
- We apply it to communication networks and propose a method for analysis and identification of simple packet data networks.

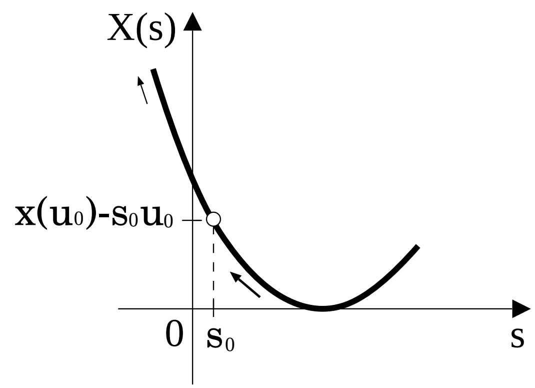
## LEGENDRE TRANSFORM

Legendre transform $\mathcal{L}[x(u)](s)$ of the function $x(u)$ that is concave or convex and has an invertible derivative:

$$X(s) := \mathcal{L}[x(u)](s) = x(u^*) - su^*, \quad \text{where } s = \frac{dx}{du}(u^*)$$

Legendre transform of a smooth concave function.

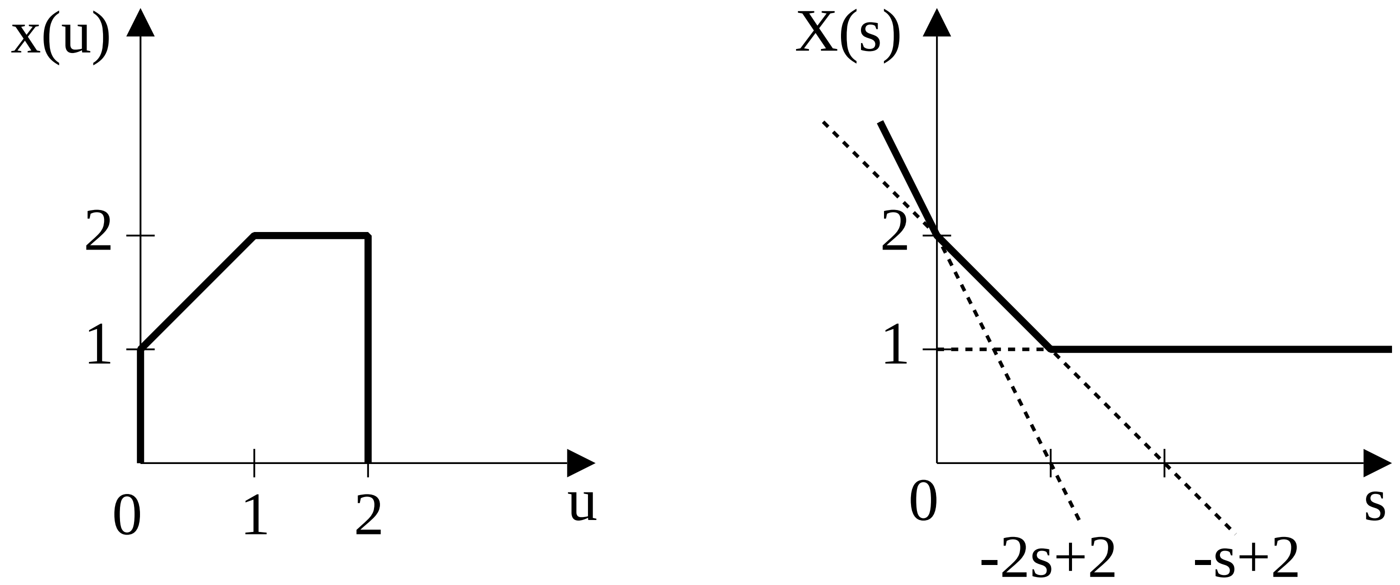



- Maximum of $x(u)$ corresponds to the intercepts of $X(s)$.
- Minimum of $X(s)$ corresponds to the intercepts of $x(u)$.

## LEGENDRE TRANSFORM: extensions

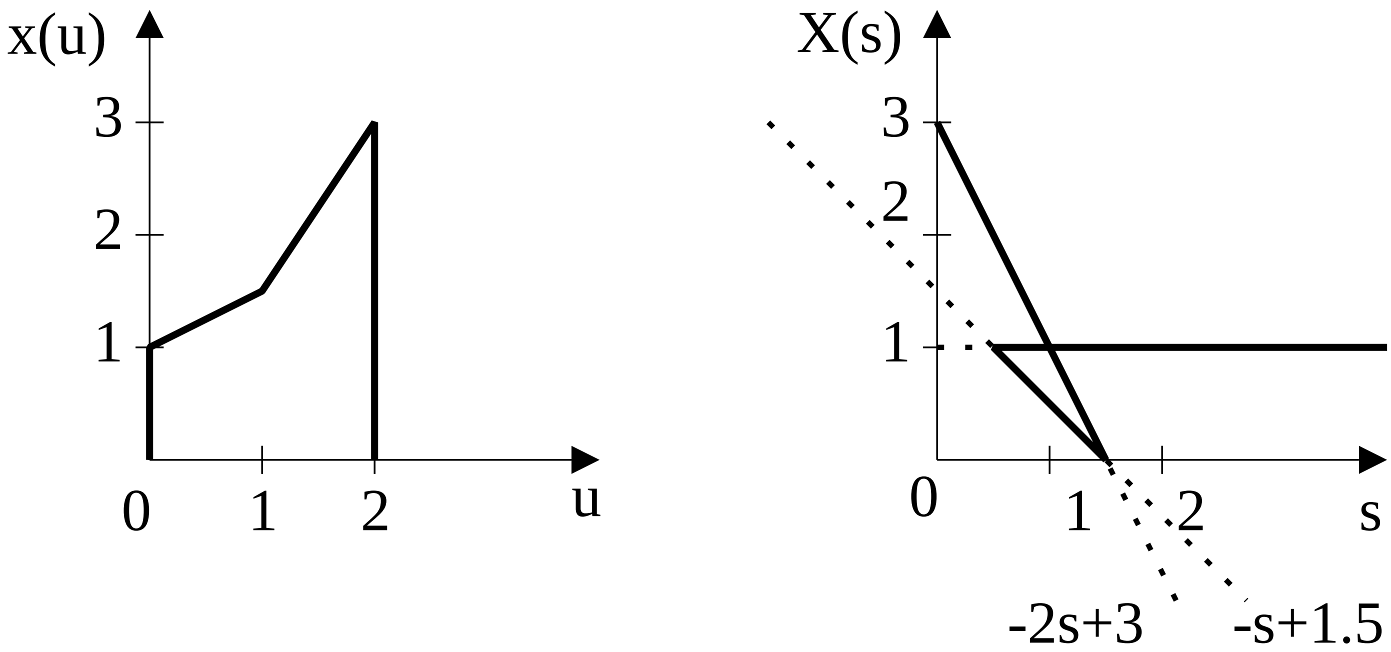
Non-smooth functions:



## LEGENDRE TRANSFORM: extensions

Non-convex/non-concave functions:

$$x(u) := \mathcal{L}^{-1}[X(s)](u) = \left\{ X(s^*) + us^* \;\middle|\; \frac{dX}{ds}(s^*) = u \right\}$$



# NETWORK SYSTEM DESCRIBED BY MAX-PLUS ALGEBRA

- In max-plus algebra, communication networks can be described as linear time-invariant systems.
- We consider a single input/single output system:
  - input of the system is denoted by the time instance $x(k)$ when the $k$-th packet is sent from a source.
  - output of the system is the time instance $y(k)$ when the $k$-th packet reaches the destination.
- Both the input and the output are non-decreasing functions.

# NETWORK SYSTEM DESCRIBED BY MAX-PLUS ALGEBRA

The output is:

$$y(k) = \bigoplus_{i=-\infty}^{k} h(k-i) \otimes x(i) = \max_{0 \le i \le k} \{h(k-i) + x(i)\},$$

where $x(k) = h(k) = -\infty$ for $k < 0$

The response characteristic $h(k)$ is:

- protocol and network dependent
- dependent on the previous state of the network.

The Legendre transform of $y(k)$ is:

$$\mathcal{L}[y](s) = \mathcal{L}[h](s) + \mathcal{L}[x](s) = H(s) + X(s),$$

where $H(s)$ and $X(s)$ denote the Legendre transform of the set $\{h(k)\}$ and $\{x(k)\}$, respectively.

## SIMPLE COMMUNICATION NETWORK

Consider a network with a transfer characteristic:

$$H(s) := \mathcal{L}[h](s) = \begin{cases} d & \text{if} \quad s \geq 1/w \\ \infty & \text{if} \quad s < 1/w \end{cases}$$

Constants $d$ and $w$ correspond to the minimum packet delay and the maximum throughput in the network, respectively.

## SIMPLIFIED METHOD

A simplified method to obtain the output $\bar{Y}(s) = \mathcal{L}[\bar{y}(u)]$, where $\bar{y}(u)$ denotes the piecewise linear interpolation of the set $\{y(k)\}$:

- Find the piecewise linear interpolation of the input set $\{x(k)\}$, denoted by $\bar{x}(u)$.
- Let $\bar{X}(s)$, $\dot{X}_k(s)$, and $\dot{Y}_k(s)$ denote the Legendre transform of $\bar{x}(u)$, and the $k$-th input $x(k)$ and output $y(k)$, respectively.
- Assume that $x(0) = 0$, $\dot{Y}_0(s) = H(\infty)$, and $s_{-1} = \infty$, and that $\dot{Y}_{k-1}(s)$ is known.

## ALGORITHM

Calculate $\dot{Y}_k(s)$ using the following algorithm:

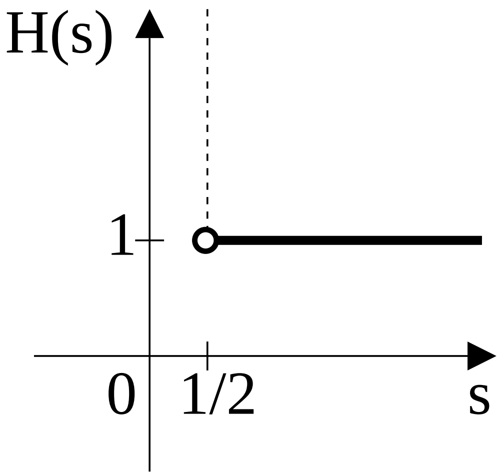
1. Calculate $\dot{Y}'_k(s) = \dot{X}_k(s) + H(s)$.
2. Find the intersection of $\dot{Y}'_k(s)$ and $\dot{Y}_{k-1}(s)$.
3. Denoted by $s_{k-1}$ the value $s$ of the intersection of $\dot{Y}'_k(s)$ and $\dot{Y}_{k-1}(s)$.
4. Obtain $\dot{Y}_k(s)$ that is parallel to $\dot{X}_k(s)$ and passes through the intersection point.

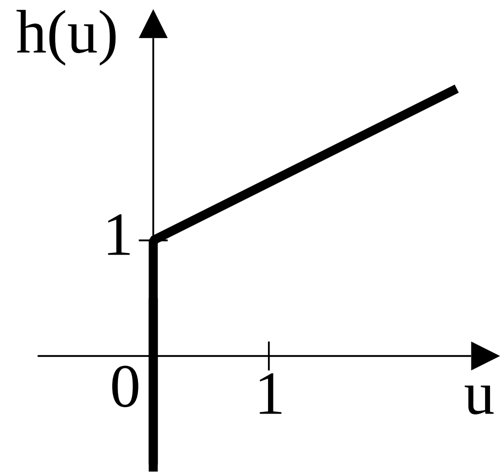
Hence, we obtain $\dot{Y}_k(s)$ from $\dot{Y}_{k-1}(s)$. $\bar{Y}(s)$ is the union of those $\dot{Y}_k(\mathrm{s})$. Its domain is bounded by $[s_{k-1}, s_k]$.

The output $\bar{y}(u)$ can be calculated using the inverse Legendre transform $\mathcal{L}^{-1}[\bar{Y}(s)](u)$.

## NETWORK TRANSFER CHARACTERISTIC

$$H(s) = \mathcal{L}[h](s) = \begin{cases} 1 & \text{if} \quad s \geq 1/2 \\ \infty & \text{if} \quad s < 1/2 \end{cases}$$

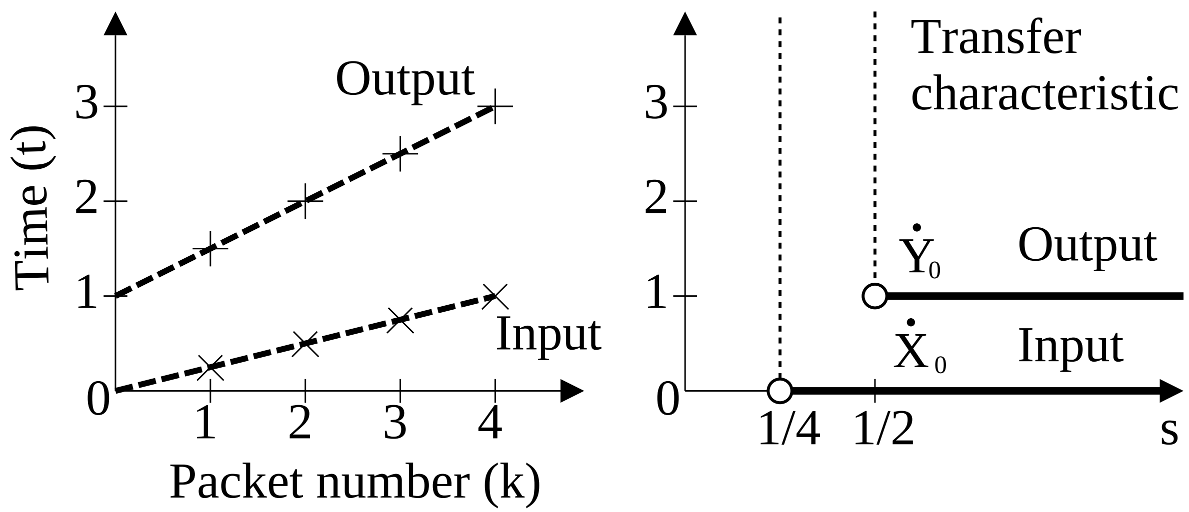



This transfer characteristic indicates that:

- network introduces a packet delay of 1 unit time
- maximum throughput is 2 packets per unit time.

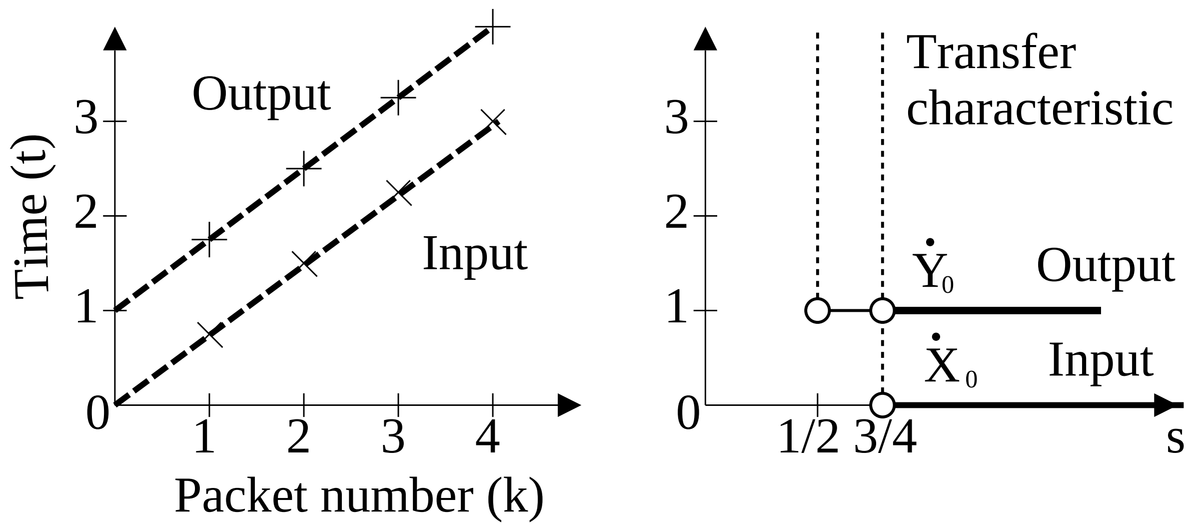
## CASE: CONGESTED NETWORK




- Network delay is 1 packet per unit time
- Maximum throughput is 2 packets per unit time
- Source sends 4 packets per unit time.

Output is the sum of the input and the transfer characteristics.

## CASE: NON-CONGESTED NETWORK




- Network delay is 1 packet per unit time
- Maximum throughput is 2 packets per unit time
- Source sends 4/3 packets per unit time.

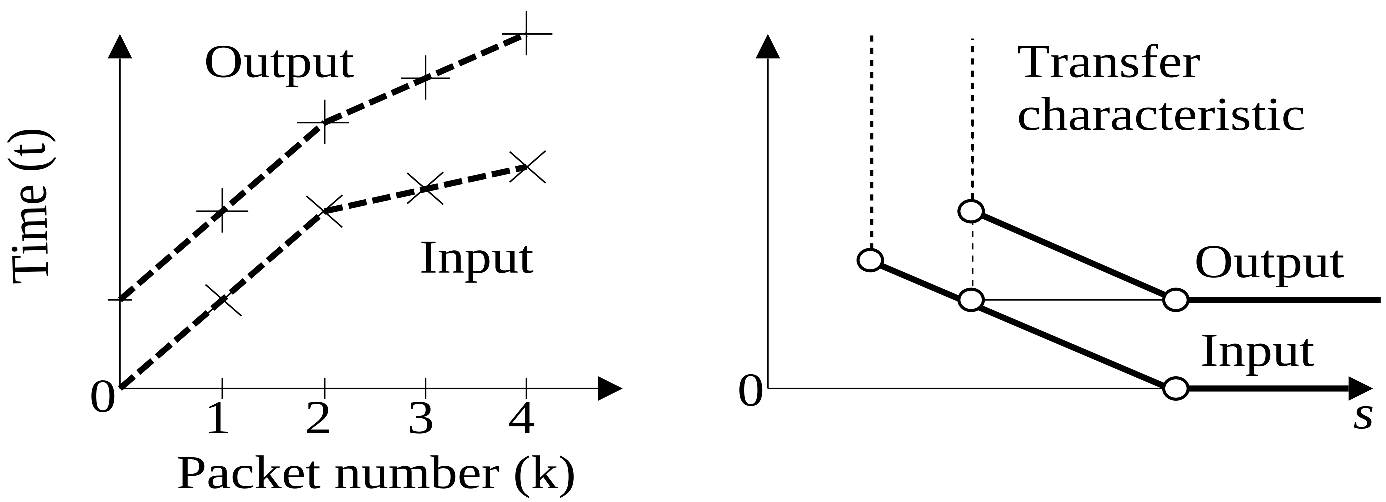
## CASE: NON-CONGESTED NETWORK

Both the congested and non-congested cases exhibit nonlinear phenomena in the sense of conventional system theory.

In max-plus algebra, we can represents the network with a unique transfer characteristic.

**VARIABLE TRAFFIC: non-congested to congested case**

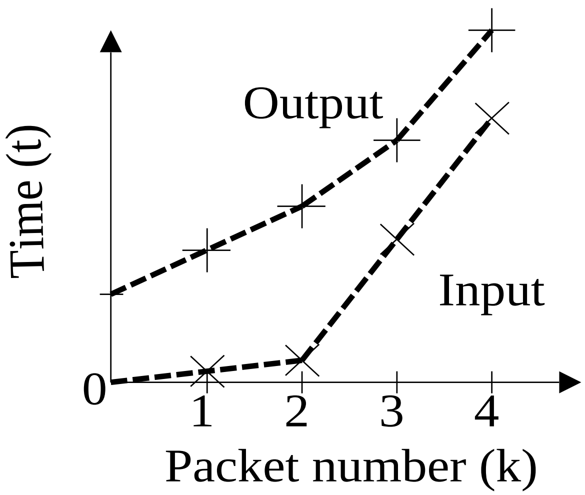
The state of the network changes from a non-congested to a congested state.

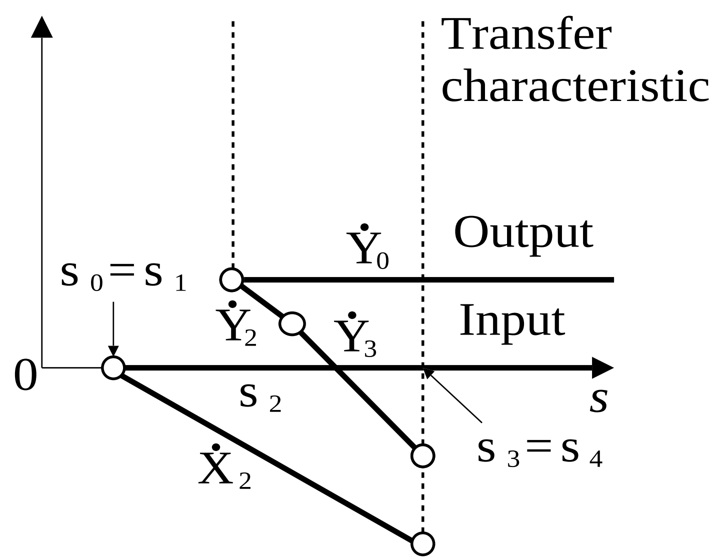



Output cannot be always calculated simply as the sum of the input and the transfer characteristic.

## VARIABLE TRAFFIC: congested to non-congested state

The state of the network changes from a congested to a non-congested state. Backlog accumulated in network during period of congestion starts to drain.




Although the output cannot be calculated simply as the sum of the input and the transfer characteristic, the phenomena can be captured using our algorithm.

## IDENTIFICATION METHOD

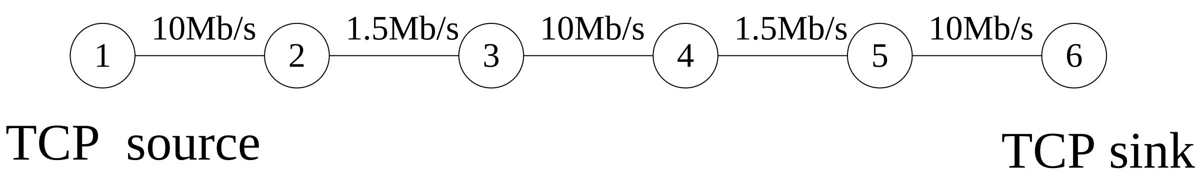
The procedure for finding the transfer characteristic of the network:

1. Calculate the piecewise linear interpolations $\bar{x}(u)$ and $\bar{y}(u)$.
2. Obtain $\bar{X}(s)$ and $\bar{Y}(s)$ by applying the Legendre transform to $\bar{x}(u)$ and $\bar{y}(u)$.
3. Obtain the transfer characteristic $H(s)$ based on the difference of $\bar{Y}(s)$ and $\bar{X}(s)$.
4. Obtain the transfer characteristic at $s_{k-1}$ as $\dot{Y}_k(s_{k-1}) - \dot{X}_k(s_{k-1})$.
5. Because $\mathcal{L}[y(k) - x(k)] = \dot{Y}_k(s) - \dot{X}_k(s)$, obtain the transfer characteristic $H(s)$ by plotting $(s_{k-1}, y(k) - x(k))$.

# IDENTIFICATION OF A SIMPLE NETWORK: ON/OFF TRAFFIC SOURCES

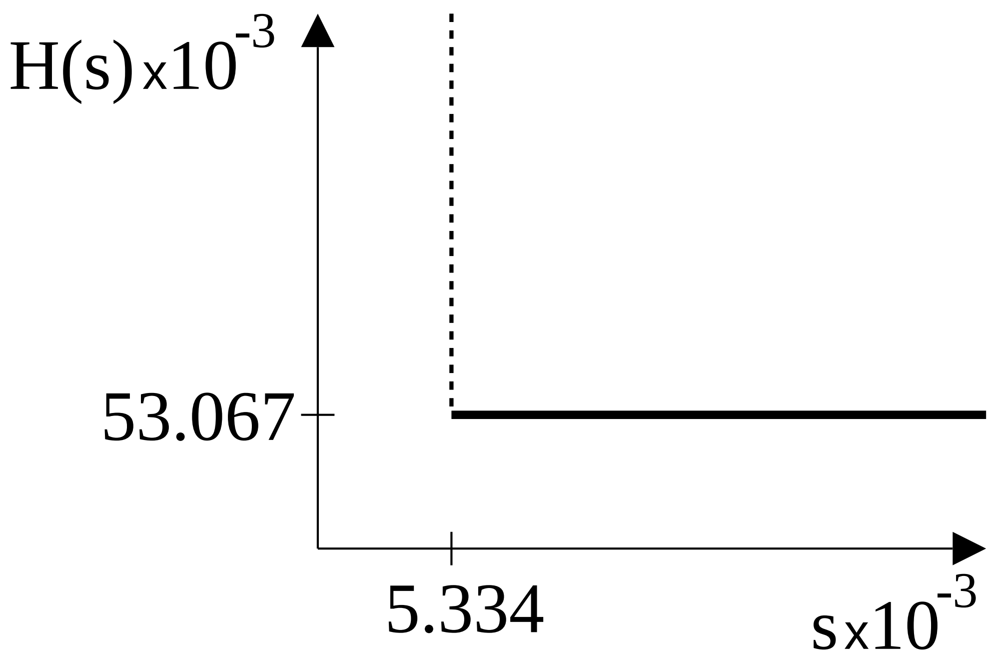
Network with:

- six nodes, FIFO queuing scheme, without buffer overflows
- transmission rates of 10, 1.5, 10, 1.5, and 10 Mbps
- links between are with infinite buffers (queues)
- TCP packet size is 1,000 bytes.



## THE SAME NETWORK WITH PARETO TRAFFIC SOURCES

Analytically identified transfer characteristic for TCP case with ON/OFF traffic trace: minimum packet delay $d = 53.067$ msec and maximum throughput $w = 187.48$ packets/sec (1.5 Mbps).



## THE SAME NETWORK WITH PARETO TRAFFIC SOURCES

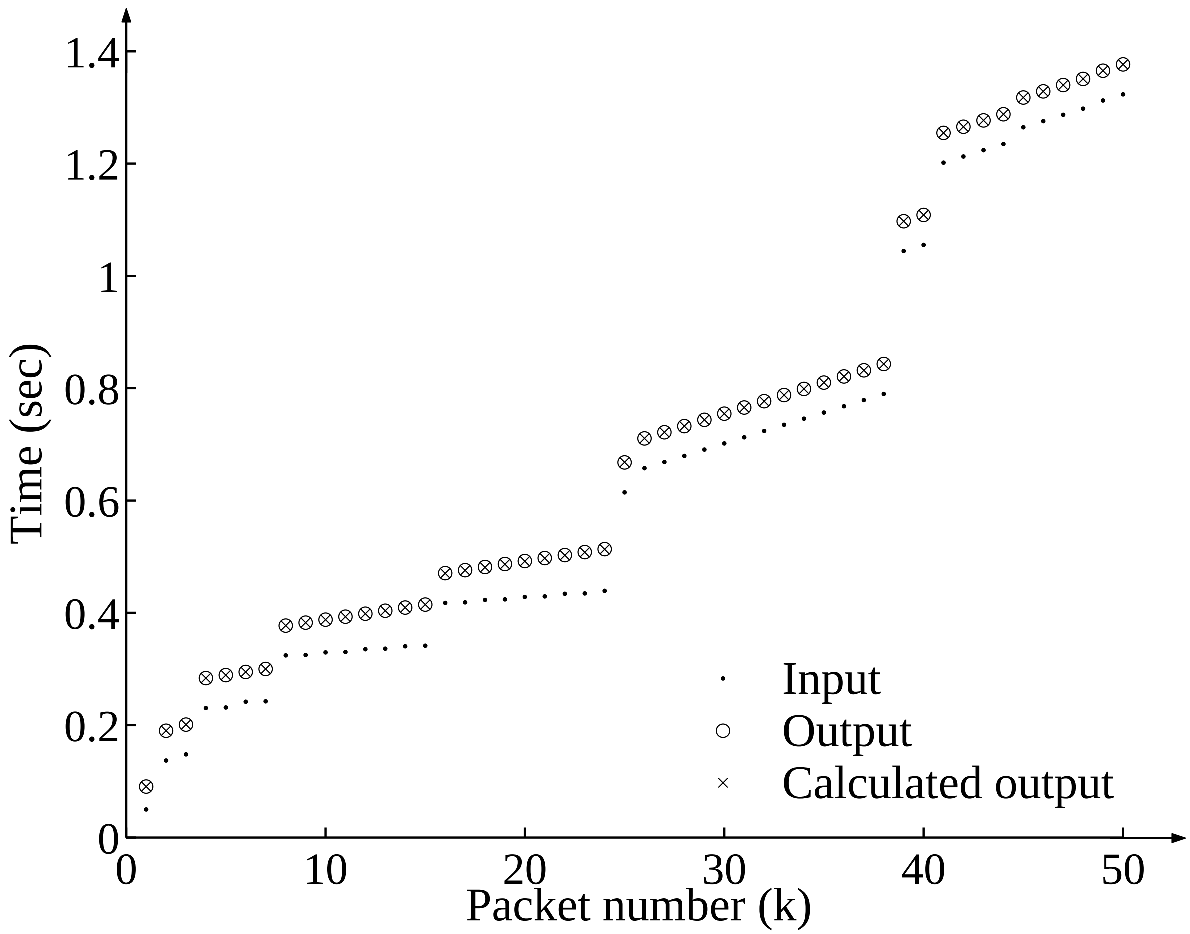
- We consider the same network with traffic that follows the Pareto distribution.
- We calculate $\tilde{y}(u)$ using the same transfer characteristic.
- The obtained transfer characteristic effectively predicts the response of the network.

## COMPARISON WITH ns-2 SIMULATION RESULTS

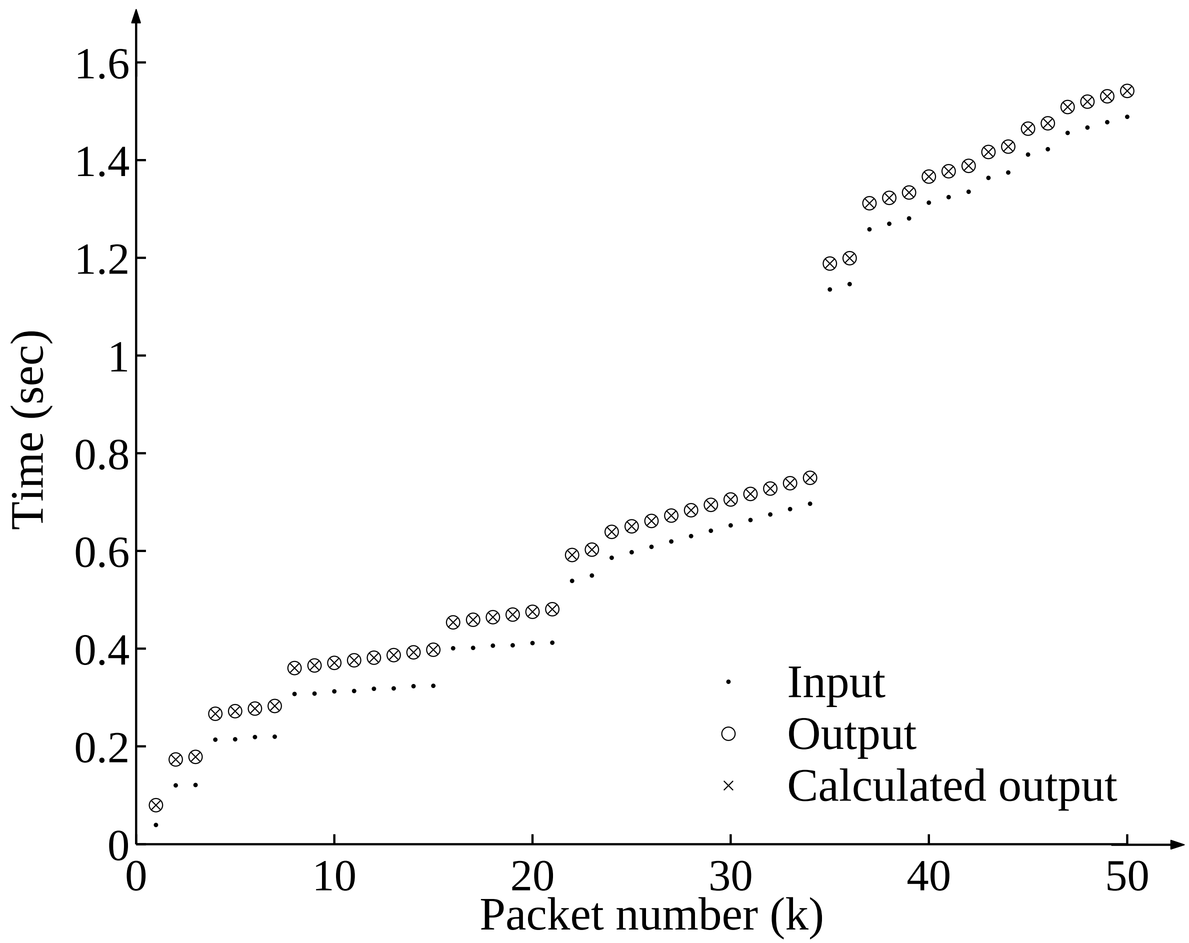
*ns-2* parameters: burst period 50 msec, idle period 50 msec, and bit rate 400 kbps, for the Pareto case: shape = 1.5.

Simulation results for a network with an ON/OFF and Pareto traffic:

# NETWORK WITH ON/OFF TRAFFIC



# NETWORK WITH PARETO TRAFFIC



## CONCLUSIONS

- We proposed an application of the Legendre transform for analysis and identification of transfer characteristic of communication networks.
- Using the Legendre transform we described the features of a communication network with a very simple transfer characteristic.
- We applied the proposed method to a simple network model and we confirmed its effectiveness using the *ns-2* network simulator.
- In the present form, the method is applicable only to simple single input/single output communication systems.

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## PUBLICATION

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