Perfect Reconstruction FIR Filter Banks and Image Compression

Description: The main focus of this assignment is to study how two-channel perfect reconstruction FIR filter banks work in image compression area and to find the advantages and disadvantages of perfect reconstruction FIR filter banks obtained by different methods.

1. 1-D Case Study
Let us design a set of filter banks which satisfy the perfect reconstruction conditions and see what we should keep in mind when designing or using those filters.

1.1 Two-channel Reconstruction Conditions
A 1-D perfect reconstruction system with two-channel filter banks is shown in Fig.1.

\[
\hat{X}(z) = T(z)X(z) + S(z)X(-z) \tag{1}
\]

where

\[
T(z) = \frac{1}{2} [H_1(z)K_1(z) + H_2(z)K_2(z)] \tag{2}
\]

\[
S(z) = \frac{1}{2} [H_1(-z)K_1(z) + H_2(-z)K_2(z)] \tag{3}
\]

In order to get perfect construction, we need \( S(z) = 0 \) that guarantees no aliasing, and \( T(z) = cz^{-n_0} \), where \( c \) is a constant and \( n_0 \) is delay, which means the reconstructed signal might be a scaled or/and delayed version of original signal, i.e.

\[
\hat{x}(n) = cx(n - n_0) \quad \text{or} \quad x(n) = \frac{1}{c} \hat{x}(n + n_0). \tag{4}
\]

In most of design methods, \( c \) and \( n_0 \) are implicit, so when we use a set of filters to build perfect reconstruction system, the post-processing, “PostP”, which is shown in Fig.1 is necessary. In other words before we use a set of filters that have satisfied perfect reconstruction conditions, we still have to calculate what \( c \) and \( n_0 \) are.
1.2 A Design Example

Let us work out a simple problem [1]. Design a set of four-tap filters that satisfy the perfect reconstruction conditions.

By the method discovered by Simth and Barnwell [2], the perfect reconstruction condition is

\[ \sum_{k=0}^{N} h_k h_{k+2n} = \delta_n \quad \text{where } N \text{ is odd and equals tap-1} \] (5)

This condition means for perfect reconstruction, the impulse response of the prototype filter is orthogonal to the twice-shifted version of itself.

Let us begin with the low pass linear phase FIR filter, which has four taps. Define

\[ H_1(z) = h_{10} + h_{11}z^{-1} + h_{12}z^{-2} + h_{13}z^{-3} \quad \text{where } h_{1j} j = 0,1,2,3 \text{ is real number} \] (6)

Based on equation (5) where \( N = 3 \), we have

\[ h_{10}^2 + h_{11}^2 + h_{12}^2 + h_{13}^2 = 1 \quad \text{for } n = 0 \]
\[ h_{10}h_{12} + h_{11}h_{13} = 0 \quad \text{for } n \neq 0 \]

Assuming that \( |h_{1k}| = |h_k| \quad \forall j, k \), we have

\[ |h_{10}| = |h_{11}| = |h_{12}| = |h_{13}| = \frac{1}{2} \quad \text{and } h_{10}h_{12} + h_{11}h_{13} = 0 \] (7)

We choose \( \left\{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\} \) as the coefficients of the low pass filter \( H_1(z) \). Of course we could choose some other values as long as they satisfy (7).

Using Equation (13.76) in [1]

\[ H_2(z) = z^{-N}H_1(-z^{-1}) \quad N = 3 \] , we can find the high-pass filter coefficients \( \{h_{2k}\} = \left\{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\} \). The magnitude of the transfer function \( H_1(z) \) and \( H_2(z) \) are shown in Fig. 2.

Note that they are mirrored and orthogonal with each other.

Using (13.75) in [1], we get

\[ K_1(z) = -H_2(-z) = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \quad K_2(z) = H_1(-z) = \left\{ \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \]

Finally, find

\[ T(z) = \frac{1}{2} [H_1(z)K_1(z) + H_2(z)K_2(z)] = \frac{1}{2} z^{-3} [H_1(z)H_1(z^{-1}) + H_1(-z)H_1(-z^{-1})] \]

Plugging in all the filter coefficients, we get \( T(z) = z^{-3} \), which means the final output signal with this set of perfect reconstruction filters is only a delayed version of input signal and no scaling happens.
1.3 Test Filters on Perfect Reconstruction System

Before we filter input signal, we need to perform a symmetric periodic extension in order to keep the number of output unchanged and meanwhile to eliminate border or edge artifacts [3]. The redundant part can be discarded at the final step, i.e. “PostP” in Fig.1.

Take a 1-D sequence with 512 samples as input signal and perform the symmetric extension, which is shown in Fig.3. Note only the part required by filter convolution is extended.

Let us take a closer look at the outputs after each operation in perfect reconstruction system. Fig.4 (a), (b) and (c) show the outputs after filtering, downsampling, upsampling and post-processing.
Fig. 4 outputs over a perfect reconstruction system

The MSE is obtained by

\[
\text{MSE} = \frac{1}{N} \sum_{i=1}^{N} (x(i) - \hat{x}(i))^2
\]

In this case \( P_{\text{error}} = 0 \), which means we get the perfect reconstruction!
1.4 Other FIR Perfect Reconstructions Filter Banks

The filters we just designed are called orthogonal filter banks. They have many nice features, such as energy conservation, identical analysis and synthesis. However there are no two-channel prefect reconstruction, orthogonal filter banks with filters being RIR, linear phase, and with real coefficients (except for the Haar filters) [4]. As shown in Fig.4 (a), nonlinear phase orthogonal filters result in non-symmetric output given a symmetric input signal. This is not a problem in this simple case, but if we want to decompose the signal again based on the result of first decomposition, we have to do the symmetric extension again. As a result the final output cannot be discarded since they are not symmetric any more [3].

Another design method produces biorthogonal FIR filter banks, which also satisfy prefect reconstruction conditions. The famous 9/7 filter and 5/3 filter are used in JPEG 2000 [3]. We plot the output of a symmetric input after 5/3 low-pass filter in Fig.5.

Another popular FIR two-channel prefect reconstruction filter banks are Quadrature Mirror Filters (QMF) [5]. They have linear phase and “almost” perfect reconstruction. Here the almost means they are aliasing free but with certain amplitude distortion or phase distortion. For example using Johnston 8-tap filters (pp. 415 in [1]) we get reconstruction error $P_{\text{error}} = 1.0446$. Comparing with the mean of input signal, which is 130.4004, this error is negligible.

2. 2-D Case Study

We are more interested in how to apply perfect reconstruction filter banks in image compression, which is a 2-D case. For simplicity, we could use separable filters to replace two-dimensional filters. In other words, we can use the 1-D filters first in one dimension, say rows, the in the other, say columns [1].

![Fig. 5 Symmetric output of a symmetric input after a linear phase low-pass filter](image-url)
2.1 Decomposition and reconstruction system

The decomposition and its corresponding reconstruction system are shown in Fig. 6. Note that they only represent the first decomposition, i.e., decompose an image into four subbands.

![Diagram of decomposition and reconstruction system](image)

(a) Decomposition structure

(b) Reconstruction structure

Fig. 6 Perfect Reconstruction system for image compression (adapted from Matlab help file)

Note that the order for the column and row is not fixed. We could first filter the image column by column then row by row, the results are the same.

2.2 Test filters on perfect reconstruction system

The reconstructions system performs the same as the 1-D case for each dimension. Therefore the symmetric periodic extension still is required before dealing with the image in order to eliminate border and edge artifacts. We use PSNR as a standard measurement of reconstruction error. It is defined as

\[
PSNR = 20 \log_{10} \left( \frac{255}{RMSE} \right)
\]  

(9)
where the root mean square error \( RMSE = \sqrt{\frac{1}{I \times J} \sum_{i=1}^{I} \sum_{j=1}^{J} (x(i, j) - \hat{x}(i, j))^2} \) is the square root of mean square error of reconstructed image. We also use error images to visualize the pixel-to-pixel errors. The error image is constructed by
\[
E(i, j) = c \times (x(i, j) - \hat{x}(i, j)) + 128
\] (10)
Here \( c \) is a constant used to adjust the errors to a gray scaled image. Since in the perfect reconstruction system the errors are very small, we use different \( c \) to different methods in order to visualize the errors.

Let us conduct some experiments by using orthogonal, biorthogonal (9/7, 5/3 filters), QMF and our designed perfect reconstruction filter banks. The results are shown in Table 1.

<table>
<thead>
<tr>
<th>Filter Name</th>
<th>Smith-Barnwell filters</th>
<th>9/7 filters</th>
<th>5/3 filters</th>
<th>Johnston filters</th>
<th>Filters we designed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design method</td>
<td>Orthogonal</td>
<td>Biorthogonal</td>
<td>Biorthogonal</td>
<td>QMF</td>
<td>Orthogonal</td>
</tr>
<tr>
<td>Filter Length(taps)</td>
<td>8</td>
<td>9 and 7</td>
<td>5 and 3</td>
<td>8</td>
<td>4</td>
</tr>
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<td>Filter Delay</td>
<td>7</td>
<td>7</td>
<td>3</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Filter Scale</td>
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<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
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<td>Perfect reconstruction</td>
<td>Almost*</td>
<td>Almost*</td>
<td>Yes</td>
<td>Almost**</td>
<td>Yes</td>
</tr>
<tr>
<td>Linear Phase</td>
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<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Energy conservation</td>
<td>Yes</td>
<td>Almost</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
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<tr>
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<td>Normal</td>
<td>No</td>
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<td>PSNR</td>
<td>73.48</td>
<td>281.84</td>
<td>Inf</td>
<td>43.17</td>
<td>Inf</td>
</tr>
</tbody>
</table>

*: the error is due to coefficients precision  
**: the error is due to design principle

The filter magnitude and phase responses are shown in Fig.7. Note that orthogonal filter banks do not have linear phase even though they can conserve energy during filter process. The reconstruction does not affected by the problem due to non-symmetric output given symmetric input since we only decompose the image once. Once we want to decompose the LL image, the whole coefficients will continue increasing and these increased coefficients cannot be discarded since they are not symmetric[3]. Thus the orthogonal filter banks are not applicable in image compression.
The property of energy compaction is visualized in Fig. 8. Our designed filters have the worst compaction, which means these filters are useless in image compression even though it can give the perfect reconstruction. Notice that the performance of energy compaction is determined by the magnitude response and sharpness of the cutoff frequency of the designed filters, rather than the design method. Comparing Fig 8(a) and Fig 8(d), the cutoff frequency for smith-Barnwell filters is much sharper than the cutoff for the Johnston filter, so the Smith-Barnwell have better energy compactions, as we have expected.

The $PSNR$ can be visualized in Fig. 9. Note that we choose different $c$ in equation (10) such that the error could be displayed. We can see that Johnston filters have the largest errors than others and 9/7 filters achieves almost perfect reconstruction. Note also that the error image of 5/3 filters shows nothing and $PSNR$ equals infinity, which means the image has been perfectly reconstructed. This perfect reconstruction is a desirable property in some cases of image compression. This is also why loseless JPEG 2000 adopts them as the analysis and synthesis filters [3]. Note also that our designed filters also can perfectly reconstruct the image, but they are not good choice since they barely have energy compaction and have nonlinear phase.
Fig. 7 the magnitude and phase response of analysis and synthesis filters

(c) S/3 filters

(d) 8-tap Johnston filters

(e) Our designed filters
Fig. 8 the decomposition outputs from different filter banks
(Note: the lines around the image are the results of symmetric extension.)
3. Conclusion
Two-channel, linear phase, FIR perfect reconstruction filter banks play important role in image compression. Both the subband coding and discrete wavelet transform employ this kind of filters. Without considering quantization errors, a set of perfect reconstruction filter banks is applicable for image compression if they have linear phase. Those filters are good if they have high energy compaction, almost energy conservation and smallest reconstruction errors. The 9/7 filters and 5/3 filters are two of good candidates.

4. Reference