Underdamped Second Order Systems

- Underdamped case results in complex numbers
- This generates a decaying oscillating case.
- Consider a case of the RLC circuit below
- Assume the Capacitor is initially charged to 10 V
- What happens is C's voltage is creates current
- That current transfers energy in the inductor L
- Energy is lost by the resistors R
- Eventually C's voltage drops below L's
- Current flow changes direction
- Inductor now transfers energy back to C
- C and L exchange energy, R losses it
- This energy loss is called Damping
Underdamped RLC circuit Equations

• continuing with the simple RLC circuit

• Recall the differential equation
• also called the "homogeneous equation"

\[ 0 = L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i \]

• Thus for the characteristic equation

\[ 0 = s^2 + s \frac{R}{L} + \frac{1}{LC} \]

• for Natural Response want initial conditions
• but no driving voltage or current applied thereafter

• The general solution is:

\[ s = -\frac{R}{2L} \pm \left[ \left( \frac{R}{2L} \right)^2 - \frac{1}{LC} \right]^{1/2} \]

\[ L = 5 \text{ mH} \]

\[ R = 10 \Omega \]
Underdamped RLC circuit Equations Con’d

- For underdamped the descriminant $< 0$

$$D = \left[ \left( \frac{R}{2L} \right)^2 - \frac{1}{LC} \right] = \left[ \left( \frac{10}{2 \times 0.005} \right)^2 - \frac{1}{0.005 \times 2 \times 10^{-6}} \right]$$

$$= 10^6 - 10^8 = -9.9 \times 10^7$$
Underdamped Second Order Systems Con'd

- Define several terms from the equations
- the damping factor is:

\[ \alpha = \frac{R}{2L} \]

- The Natural angular frequency of the circuit

\[ \omega_n^2 = \frac{1}{LC} \]

- the Natural frequency is that when no damping

- the damped frequency is:

\[ \omega^2 = \left[ \left( \frac{R}{2L} \right)^2 - \frac{1}{LC} \right] = \omega_n^2 - \alpha^2 \]

- Then can define the solutions

\[ s_1 = -\alpha + j\omega \quad s_2 = -\alpha - j\omega \]

- The combined solution is

\[ i(t) = A_1 \exp((-\alpha + j\omega)t) + A_2 \exp((-\alpha - j\omega)t) \]

- The A's are constants set by the initial conditions
Example Underdamped Second Order Circuit Con’d

- for the example case L = 5 mH, C = 2 µF, R = 10 ohms
- also assume C is charged to 10 V at t = 0
- the damping factor is:
  \[ \alpha = \frac{R}{2L} = \frac{10}{2 \times 0.005} = 10^3 \]

- The Natural angular frequency of the circuit
  \[ \omega_n^2 = \frac{1}{LC} = \frac{1}{0.005 \times 2 \times 10^{-6}} = 10^8 \]

- the damped frequency is:
  \[ \omega^2 = \omega_n^2 - \alpha^2 = 10^8 - 10^6 = 9.9 \times 10^7 \]
  \[ \omega = 9.95 \times 10^3 \]

- and the period is
  \[ T = \frac{2\pi}{\omega} = \frac{2\pi}{9.95 \times 10^3} = 6.31 \times 10^{-4} \text{ sec.} \]

- The combined solution is
  \[ i(t) = A_1 \exp([-\alpha + j\omega]t) + A_2 \exp([-\alpha - j\omega]t) \]

![Circuit Diagram](image)
Initial Underdamped Second Order Systems Con’d

- for the example case $L = 5$ mH, $C = 2$ mF, $R = 10$ ohms
- Since $L$ acts as an open initially then $i(0) = 0$, thus
  \[ i(0) = A_1 + A_2 = 0 \]

- because $\exp(0) = 1$, thus
  \[ A_2 = -A_1 \]

- Hence
  \[ i(t) = A_1 \exp(-\alpha t) [\exp(j\omega t) - \exp(-j\omega t)] \]
  \[ = A_1 \exp(-\alpha t) 2j \sin(\omega t) \]

because
  \[ \sin(\theta) = \frac{1}{2j} [\exp(j\theta) - \exp(-j\theta)] \]
Initial Underdamped Second Order Systems Con’d

- since the inductor acts open at time zero

\[ V_c = L \frac{di}{dt} = V_L \]

\[ \frac{di(t=0)}{dt} = A_1 2j[-\alpha \exp(-\alpha t) \sin(\omega t) + \omega \exp(-\alpha t) \cos(\omega t)] \]

\[ = 2\omega j A_1 \frac{di(t=0)}{dt} \]

because \( \sin(0) = 0 \) and \( \cos(0) = 1 \).

\[ \frac{V_c}{L} = 2j \omega A_1 \]

\[ A_1 = \frac{V_c}{2j \omega L} = \frac{10}{2j \times 9.95 \times 10^3 \times 0.005} = \frac{0.201}{2j} A \]

The 2j term is eliminates that from the sin function

\[ i(t) = 0.201 \exp(-10^3 t) \sin(9.95 \times 10^3 t) \, A \]
\[ I(t) = 0.2 + 0.8 \exp(-10t) \]