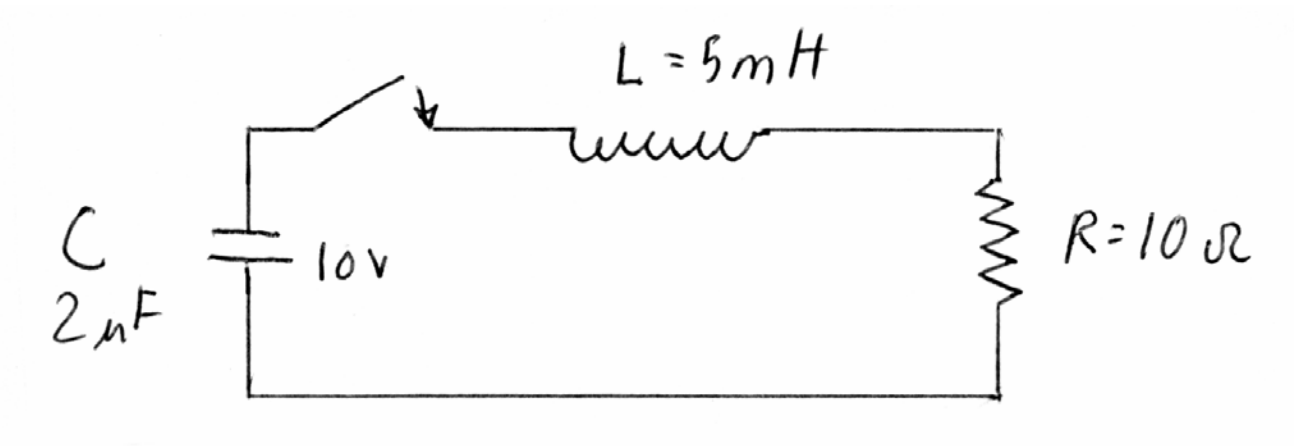
Underdamped Second Order Systems

- Underdamped case results in complex numbers
- This generates a decaying oscillating case.
- Consider a case of the RLC circuit below
- Assume the Capacitor is initially charged to 10 V
- What happens is C's voltage is creates current
- That current transfers energy in the inductor L
- Energy is lost by the resistors R
- Eventually C's voltage drops below L's
- Current flow changes direction
- Inductor now transfers energy back to C
- C and L exchange energy, R losses it
- This energy loss is called Damping



* Simple RLC circuit, underdamping, C charged initially
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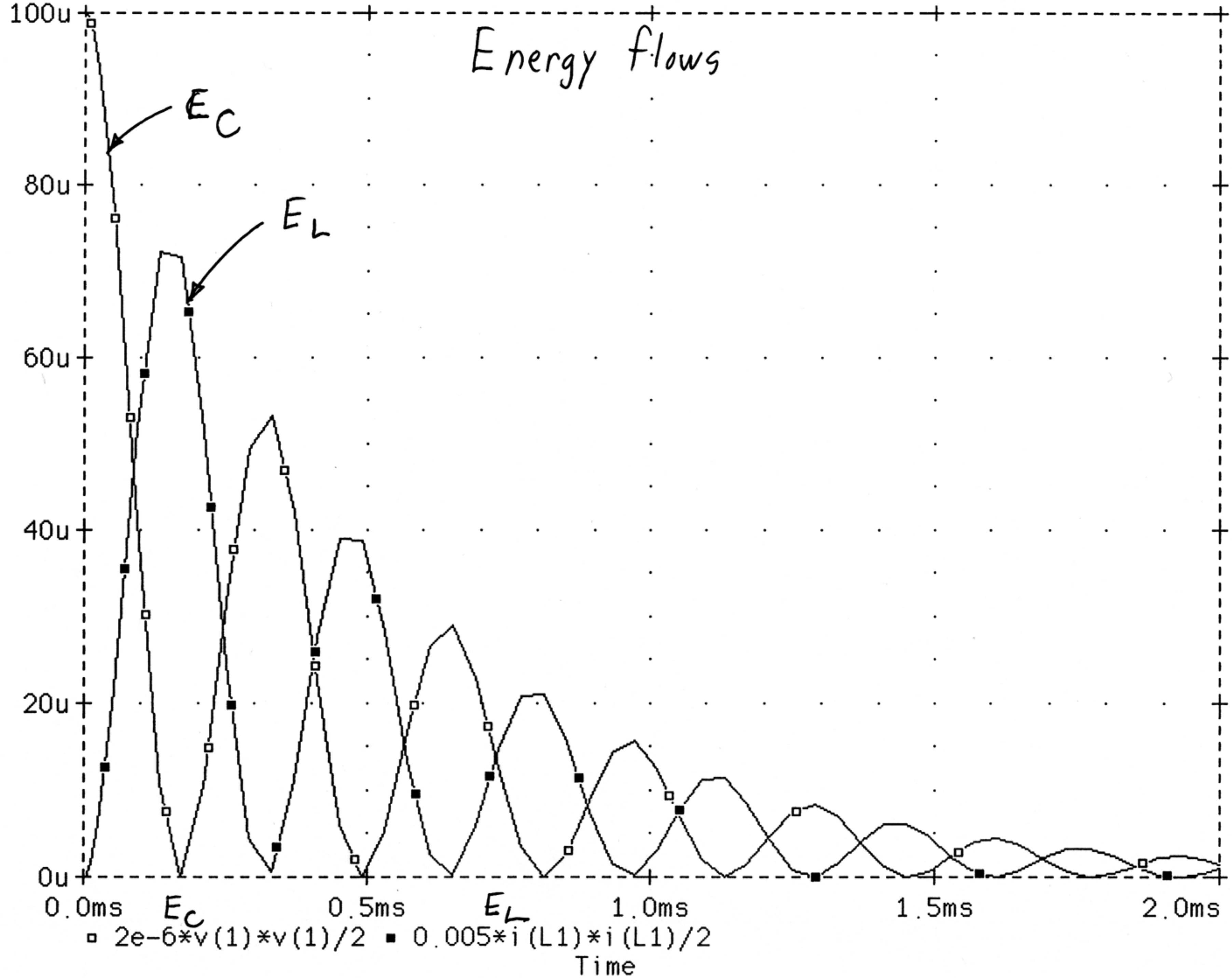


Underdamped RLC circuit Equations

- continuing with the simple RLC circuit

- Recall the differential equation
- also called the "homogeneous equation"

$$0 = L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i$$

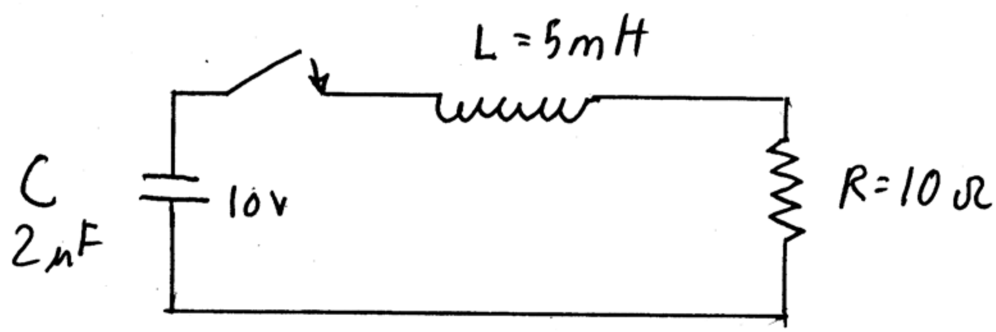
- Thus for the characteristic equation

$$0 = s^2 + s\frac{R}{L} + \frac{1}{LC}$$

- for Natural Response want initial conditions
- but no driving voltage or current applied thereafter

- The general solution is:

$$s = -\frac{R}{2L} \pm \left[\left(\frac{R}{2L} \right)^2 - \frac{1}{LC} \right]^{1/2}$$



Underdamped RLC circuit Equations Con'd

- For underdamped the descriminant < 0

$$D = \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right] = \left[\left(\frac{10}{2x0.005}\right)^2 - \frac{1}{0.005x2x10^{-6}}\right]$$

$$= 10^6 - 10^8 = -9.9x10^7$$

Underdamped Second Order Systems Con'd

- Define several terms from the equations
- the damping factor is:

$$\alpha = \frac{R}{2L}$$

- The Natural angular frequency of the circuit

$$\omega_n^2 = \frac{1}{LC}$$

- the Natural frequency is that when no damping

- the damped frequency is:

$$\omega^2 = \left[\left(\frac{R}{2L} \right)^2 - \frac{1}{LC} \right] = \omega_n^2 - \alpha^2$$

- Then can define the solutions

$$s_1 = -\alpha + j\omega \qquad s_2 = -\alpha - j\omega$$

- The combined solution is

$$i(t) = A_1 \exp([-\alpha + j\omega]t) + A_2 \exp([-\alpha - j\omega]t)$$

- The A's are constants set by the initial conditions

Example Underdamped Second Order Circuit Con'd

- for the example case L = 5 mH, C = 2 μF, R = 10 ohms
- also assume C is charged to 10 V at t = 0
- the damping factor is:

$$\alpha = \frac{R}{2L} = \frac{10}{2x0.005} = 10^3$$

- The Natural angular frequency of the circuit

$$\omega_n^2 = \frac{1}{LC} = \frac{1}{0.005x2x10^{-6}} = 10^8$$

- the damped frequency is:

$$\omega^2 = \omega_n^2 - \alpha^2 = 10^8 - 10^6 = 9.9x10^7$$

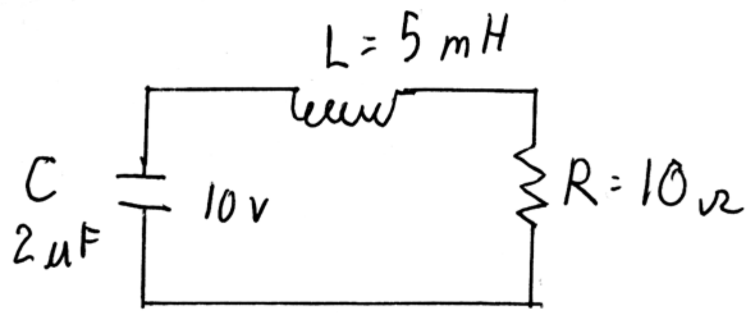
$$\omega = 9.95x10^3$$

- and the period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{9.95x10^3} = 6.31x10^{-4}\ sec.$$

- The combined solution is

$$i(t) = A_1 \exp([-\alpha + j\omega]t) + A_2 \exp([-\alpha - j\omega]t)$$



Initial Underdamped Second Order Systems Con'd

- for the example case L = 5 mH, C = 2 mF, R = 10 ohms
- Since L acts as an open initially then i(0) = 0, thus

$$i(0) = A_1 + A_2 = 0$$

- because exp(0) = 1, thus

$$A_2 = -A_1$$

- Hence

$$i(t) = A_1 \exp(-\alpha t)[\exp(j\omega t) - \exp(-j\omega t)]$$

$$= A_1 \exp(-\alpha t) 2j \sin(\omega t)$$

because

$$\sin(\theta) = \frac{1}{2j}[\exp(j\theta) - \exp(-j\theta)]$$

Initial Underdamped Second Order Systems Con'd

- since the inductor acts open at time zero

$$V_c = L\frac{di}{dt} = V_L$$

$$\frac{di(t=0)}{dt} = A_1 2j[-\alpha \exp(-\alpha t)\sin(\omega t) + \omega\exp(-\alpha t)\cos(\omega t)$$

$$= 2\omega j A_1 = \frac{di(t=0)}{dt}$$

because sin(0) = 0 and cos(0) = 1.

$$\frac{V_c}{L} = 2j\omega A_1$$

$$A_1 = \frac{V_c}{2j\omega L} = \frac{10}{2jx9.95x10^3x0.005} = \frac{0.201}{2j} A$$

The 2j term is eliminates that from the sin function

$$i(t) = 0.201\exp(-10^3 t)\sin(9.95x10^3 t)\ A$$

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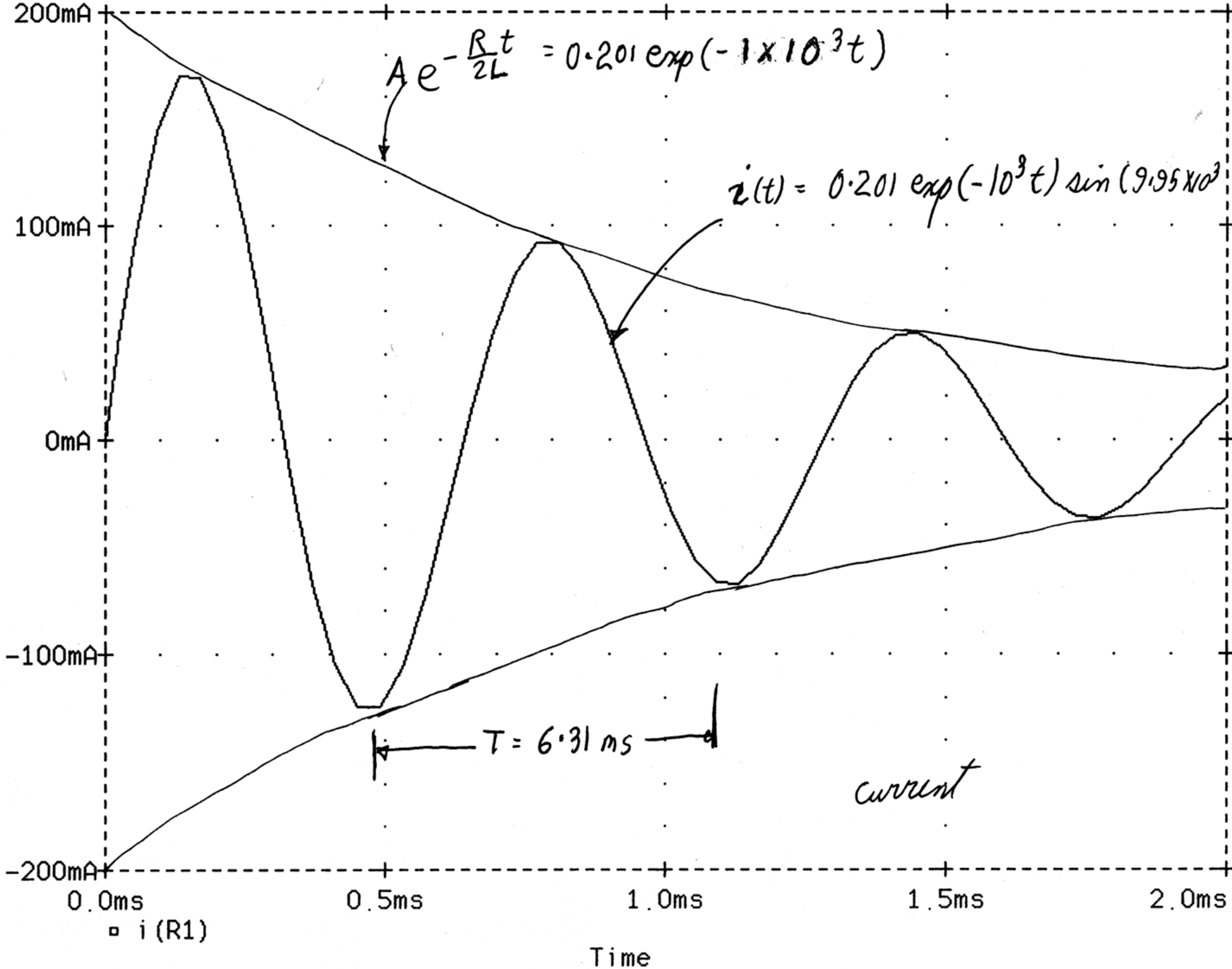

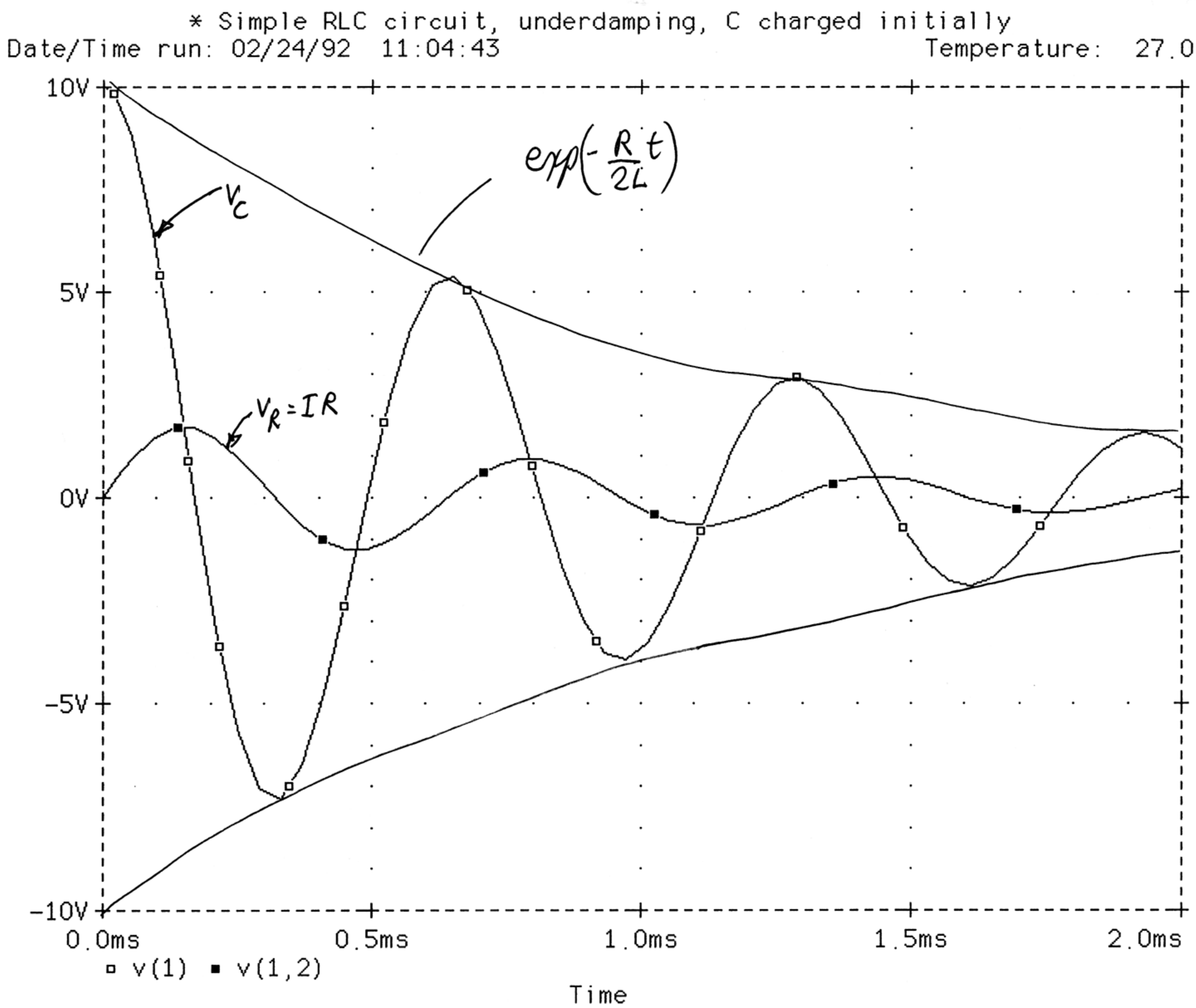
* Simple RLC circuit, underdamping, C charged initially
Date/Time run: 02/24/92 11:04:43
Temperature: 27.0
10V
5V
0V
-5V
-10V
0.0ms
0.5ms
1.0ms
1.5ms
2.0ms
v(1)
v(1,2)
Time
V_C
$V_R = IR$
$exp\left(-\frac{R}{2L}t\right)$

* Simple RLC circuit, underdamping, C charged initially

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