Parallel RLC Second Order Systems

- Consider a parallel RLC
- Switch at t=0 applies a current source
- For parallel will use KCL
- Proceeding just as for series but now in voltage

(1) Using KCL to write the equations:

$$C\frac{di}{dt}+\frac{v}{R}+\frac{1}{L}\int_0^t vdt = I_0$$

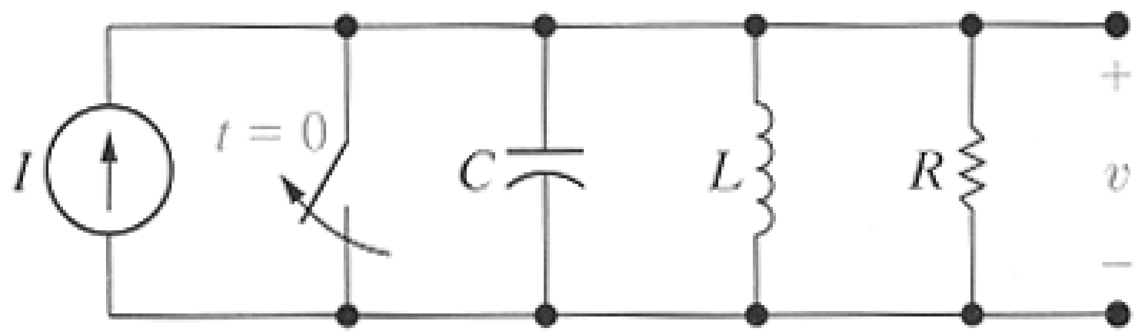
(2) Want full differential equation

- Differentiating with respect to time

$$C\frac{d^2v}{dt^2}+\frac{1}{R}\frac{dv}{dt}+\frac{1}{L}v = 0$$

(3) This is the differential equation of second order

- Second order equations involve 2nd order derivatives



Solving the Second Order Systems Parallel RLC

- Continuing with the simple parallel RLC circuit as with the series (4) Make the assumption that solutions are of the exponential form:

$$i(t) = A\,exp(st)$$

- Where A and s are constants of integration.
- Then substituting into the differential equation

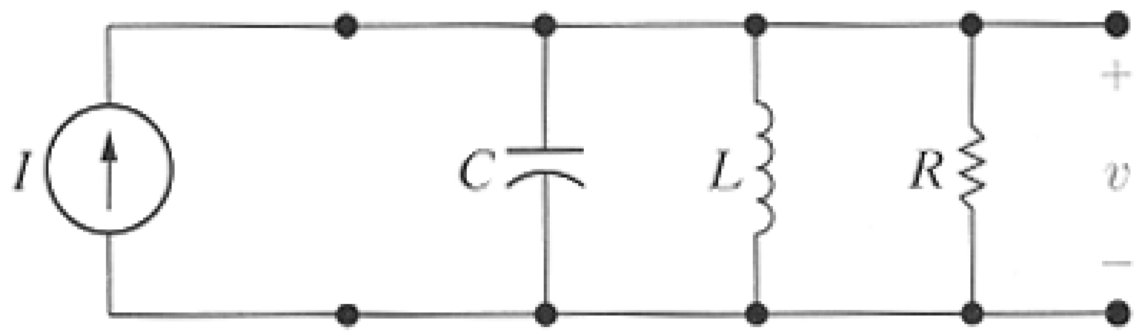
$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

$$Cs^2 A \exp(st) + \frac{1}{R} sA \exp(st) + \frac{A}{L}\exp(st) = 0$$

- Dividing out the exponential for the characteristic equation

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

- Giving the Homogeneous equation
- Get the 3 same types of solutions but now in voltage
- Just parameters are going to be different



General Solution Parallel RLC

- Solving the homogeneous quadratic as before

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

- The general solution is:

$$s = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

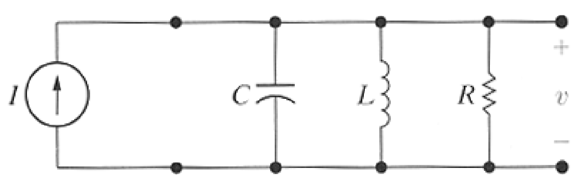
- Note the difference from the series RLC

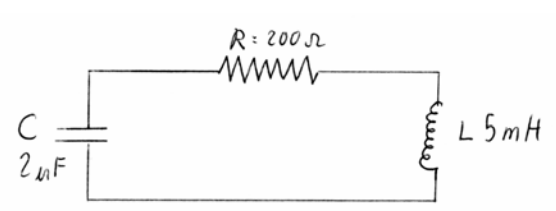
$$s_{series} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

- Note the difference is in the damping term first term
- Again type of solution is set by the Descriminant

$$D = \left[\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}\right]$$

- Recall RC is the time constant of the resistor capacitor circuit





3 solutions of the Parallel RLC

- What the **Descriminant** represents is about energy flows

$$D = \left[\left(\frac{1}{2RC} \right)^2 - \frac{1}{LC} \right]$$

- Again how fast is energy transferred from the L to the C
- How fast is energy lost to the resistor
- Get the same three cases & general equations set by D
- D > 0 : roots real and unequal: overdamped case
- D = 0 : roots real and equal: critically damped case
- D < 0 : roots complex and unequal: underdamped case
- Now the damping term changes

$$\alpha_{parallel} = \frac{1}{2RC}$$

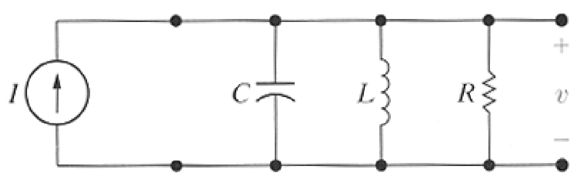
- For the series RLC it was

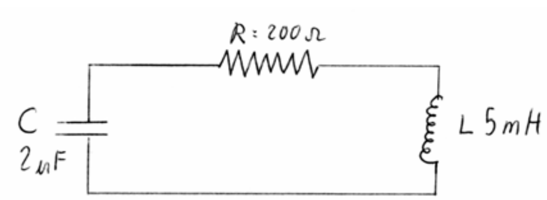
$$\alpha_{series} = \frac{R}{2L}$$

- Recall τ=RC for the resistor capacitor circuit
- While $\tau = \frac{R}{L}$ for the resistor inductor circuit
- The natural frequency (underdamped) stays the same

$$\omega_n = \frac{1}{\sqrt{LC}}$$

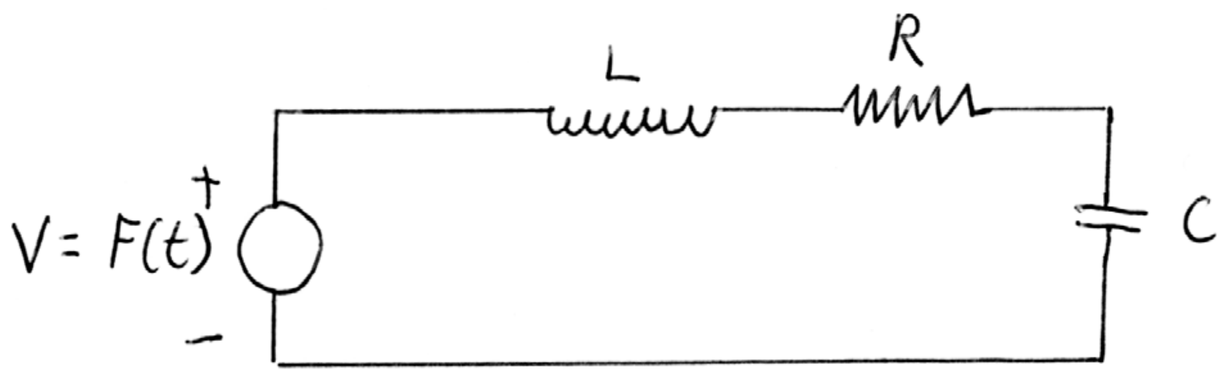
The difference is in the solutions created by the initial conditions





Forced Response & RL, RC and RLC Combination

- Natural Response: energy stored then decays
- Forced Response: voltage/current applied
- Forcing function can be anything
- Typical types are steps or sine functions
- Step response: called complete response in book
- Step involves both natural and forced response
- Forced response (Book): after steady state reached
- forced response: when forcing function applied.
- Forcing function: any applied V or I
- Most important case simple AC response



Forced Response

- How does a circuit act to a driving V or I which changes with time
- Assume this is long after the function is applied
- Problem easiest for RC & RL
- General problem difficult with RLC type
- Procedure: write the KVL or KCL laws
- Equate it to the forcing function F(t)

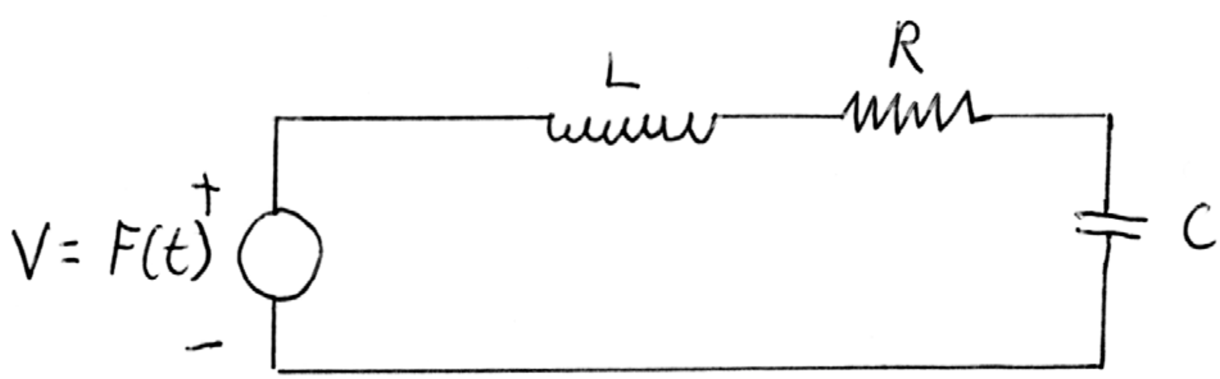
$$F(t) = \sum_{j=1}^{n} v_j$$

- Then create and solve Differential Equation

General solution difficult
Two simple Cases important:

(1) Steady V or I applied, or sudden changes at long intervals
- Just need to know how the C or L respond
- In long time C become open, L a short
- Solved as in RL and RC case
- Must have time between changes >> time constants

(2) Sinewave AC over long time
- Solved using the complex Impedance



Complete Response

- Complete response: what happens to a sudden change
- Apply a forcing function to the circuit (eg RC, RL, RLC)

- Complete response is a combination two responses

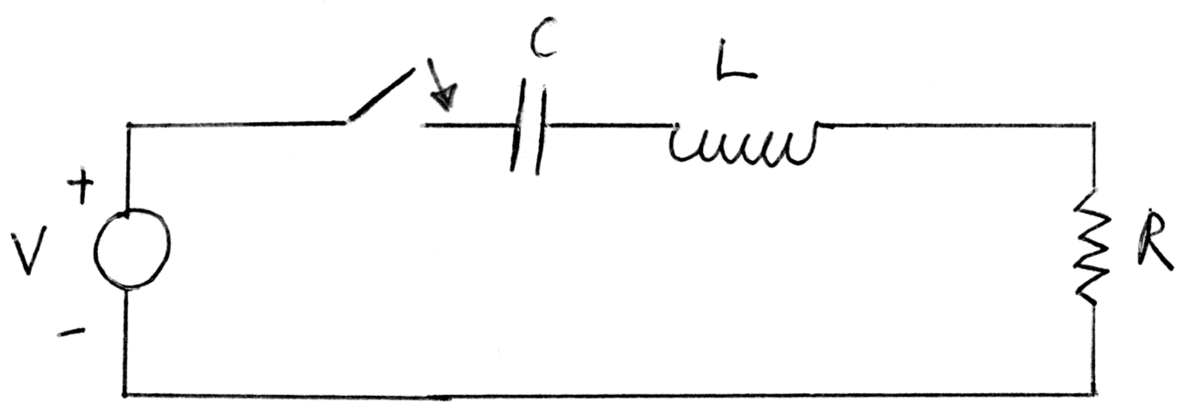
(1) First solve natural response equations
- use either differential equations
- Get the roots of the exp equations
- Or use complex impedance (coming up)

(2) Then find the long term forced response

(3) Add the two equations

$$V_{complete} = V_{natural} + V_{forced}$$

(4) Solve for the initial conditions



Example Complete Response of RL to a step

- Consider an RL circuit with a switched voltage
- This is called a voltage step

(1) From previous results the natural response is:

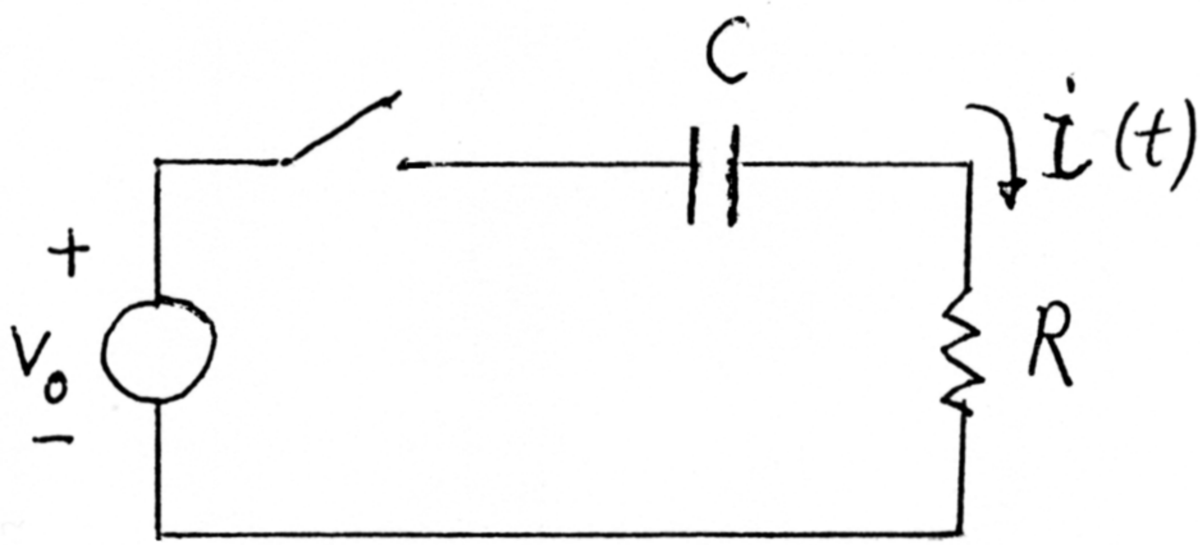
$$I(t)_{nat} = A \exp\left[-\frac{Rt}{L}\right]$$

(2) In the long term the L acts as an short

$$I(t \to \infty)_{for} = \frac{V_{in}}{R}$$

(3) Combine the equations

$$I(t) = \frac{V_{in}}{R} + A \exp\left[-\frac{Rt}{L}\right]$$



Complete Response of RL Con'd

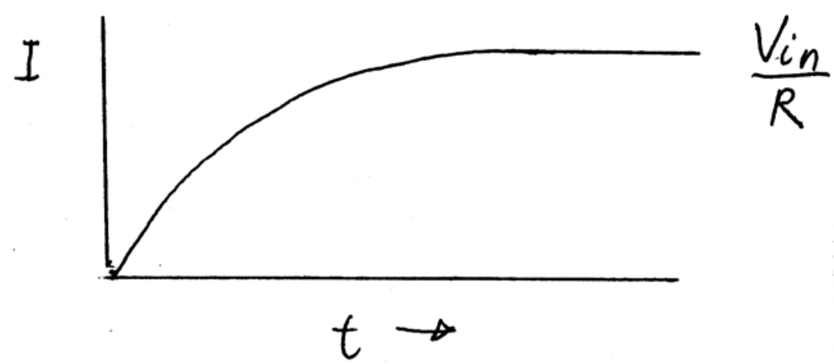
(4) Solve for inital conditions
At t=0+ no current must flow

$$0 = I(0) = \frac{V_{in}}{R} + A \ \exp\left[-\frac{R0}{L}\right]$$

$$A = -\frac{V_{in}}{R}$$

- thus the complete response equation is

$$I(t) = \frac{V_{in}}{R}\left\{1 - \exp\left[-\frac{Rt}{L}\right]\right\}$$



Complete Response In General

- General solution difficult
- Two simple Cases important:

(1) Steady V or I applied
- just need to know how the C or L respond
- in long time C become open, L a short
- solved as in RL and RC case

(2) Sinewave AC over long time
- solved using the complex Impedance (EC 10)

Complete Response of Second Order Circuits

- 2nd order complete response proceedure same as 1st
- but getting coefficients more complex

(1) First solve natural response equations
- use either differential equations
- or impedance method (next section)

(2) Then find the long term forced response
- for sudden DC changes get steady state results

(3) Add the two equations

$$I_{complete} = I_{natural} + I_{forced}$$

(4) Solve for the inital conditions.
- must use intial conditions
- For current need both

$$I(t=0+) \qquad \frac{dI(t=0+)}{dt}$$

- similar requirements for Voltage

Complete Response of series RLC to a Voltage step

- Consider an RLC circuit with a switched voltage
- Find the voltage across the capacitor

(1) From previous results for the natural response:

- in terms of the solution s of this impedance then

$$I(t)_{nat} = A_1 \, exp(s_1 t) + A_2 \exp(s_2 t)$$

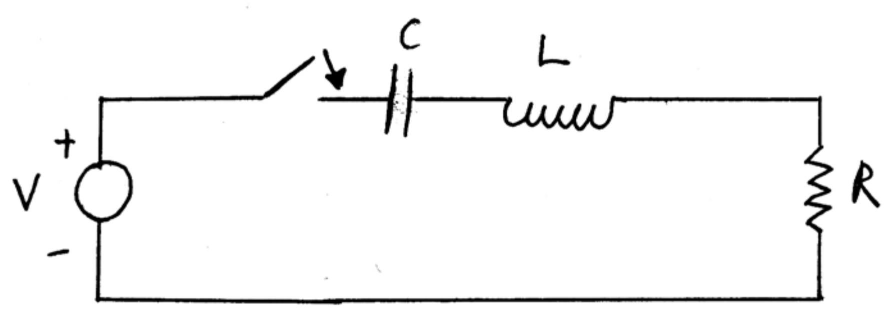
- where s' are the roots of the homogeneous equation

(2) In long term, L acts as an short, C open

$$I(t \to \infty)_{\mathrm{for}} = 0$$

(3) Combine the equations

$$I(t)_{nat} = A_1 \, exp(s_1 t) + A_2 \exp(s_2 t)$$



Complete Response of RLC series Con'd

(4) Solve for inital conditions
At t=0+ no current must flow, L open

$$0 = I(0) = A_1\, exp(s_1 0) + A_2 \exp(s_2 0)$$

$$A_1 = -A_2$$

- Also need the derivative at t=0+
- noting that L is open an C uncharged then

$$V_{in} = L\frac{di}{dt}$$

- thus:

$$\frac{V_{in}}{L} = A\,[s_1\, exp(s_1 0) - s_2 \exp(s_2 0)]$$

$$A = \frac{V_{in}}{L(s_1 - s_2)}$$

- the exact form depends on the type of roots.

Complete Response of RLC series Con'd

- if roots are complex then

$$s_1 = -\alpha + j\omega \qquad s_2 = -\alpha - j\omega$$

- and the coefficient is

$$A = \frac{V_{in}}{L(s_1 - s_2)} = \frac{V_{in}}{jL2\omega}$$

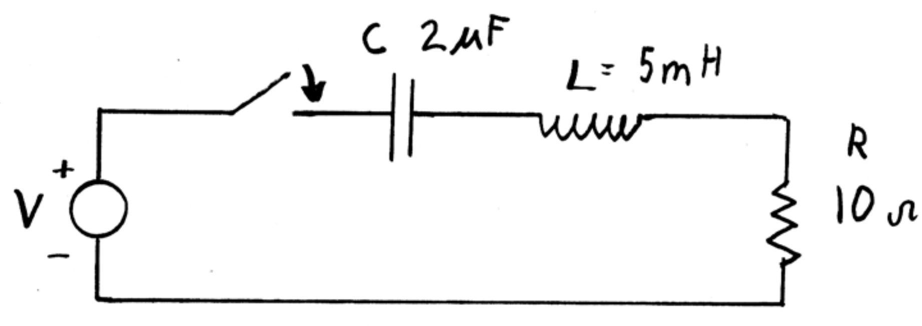
- since

$$\sin(\omega t) = \frac{1}{2j}[\exp(j\omega t) - \exp(-j\omega t)]$$

$$I(t)_{nat} = \frac{V_{in}}{L\omega}\, exp(-\alpha t)\sin(\omega t)$$

- How to solve complete responce for AC sources

- Need the complex impedance methods



Sinusoidal waves

- Most circuits involve periodic signals
- Most common are made of sinusoidal waves

$$a(t) = A_0 \cos(\omega t + \alpha)$$

Where

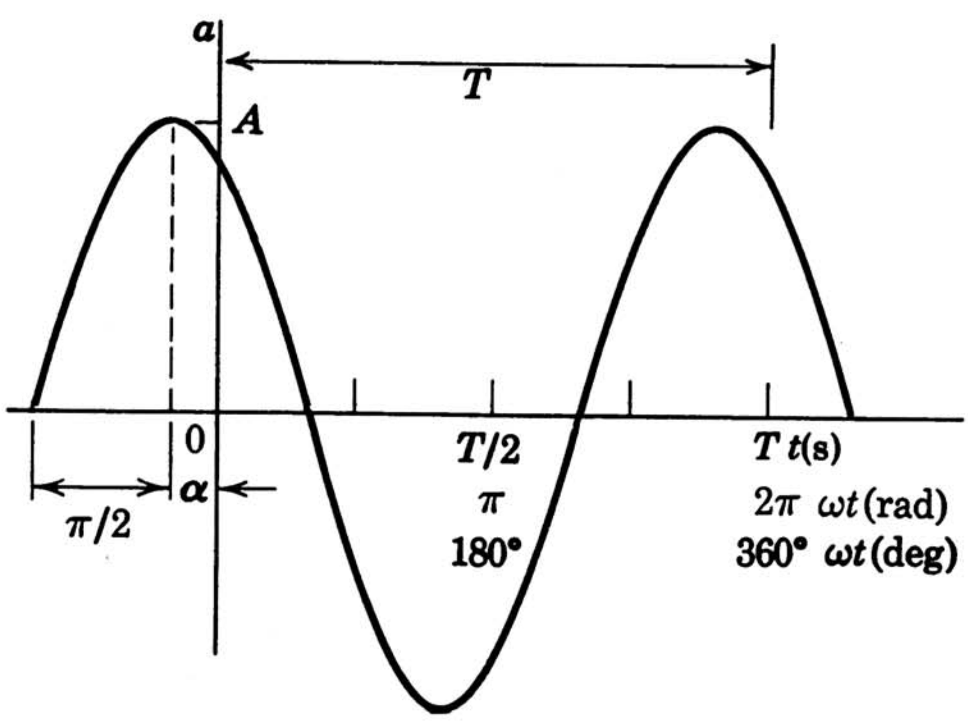
$a(t)$ = signal at time t
A_0 = max signal or amplitude
ω = (omega) radial frequency, radians/sec.
t = time in sec.
α = phase shift angle in radians (also ϕ : phi)

- radial frequency related to the regular frequency f by

$$\omega = 2\pi f$$

- The period of the wave is

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$



Sin waves

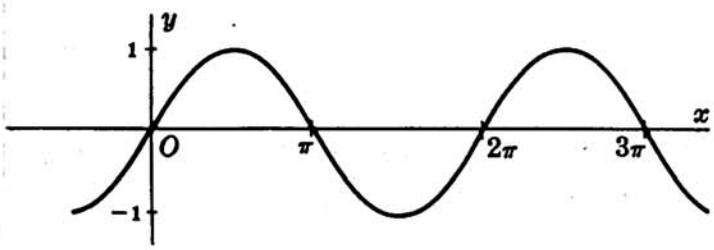
- can relate a cos wave to a sin wave via

$$a(t) = A_0 \cos(\omega t + \alpha) = A_0 \sin(\omega t + \alpha + \frac{\pi}{2})$$

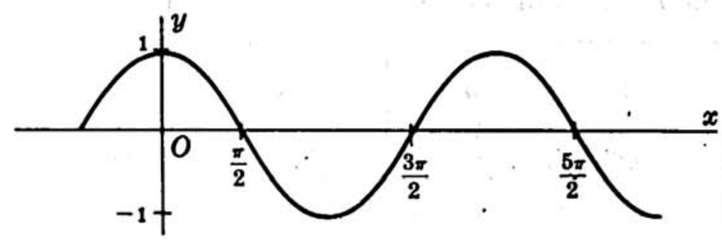
- sine wave: phase shifted by 90 degrees from cos

In each graph x is in radians.

5.22 $y = \sin x$



5.23 $y = \cos x$



Projection of Cos wave

- Useful to regard wave as a rotating vector
- rotating vector is called a phasor
- Period of rotation

$$period = \frac{2\pi}{\omega}$$

- instanteous value = projection on x axis of vector

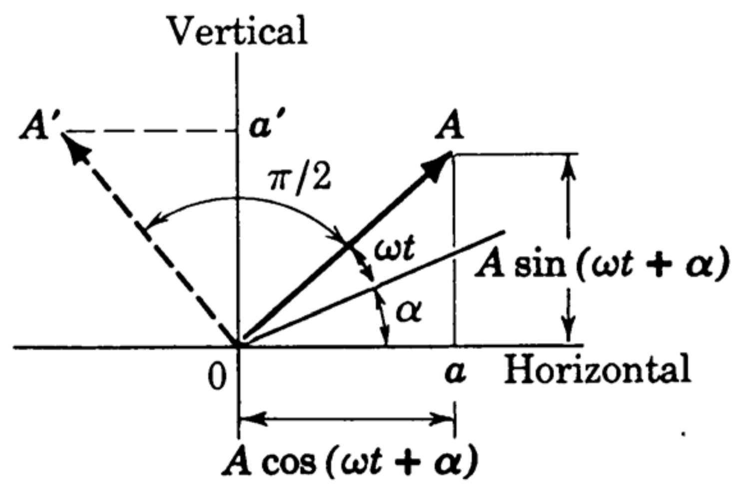


Figure 3.8 Projections of a rotating line.

- Call electrical signals of this type "Alternating Current"

Example of sinusoidal wave

- Example: a 60 Hz wave has its maximum at 2.08 msec.
- find the wave formula

- first find the radial frequency

$$\omega = 2\pi f = 2\pi 60 = 377 \; rads/sec$$

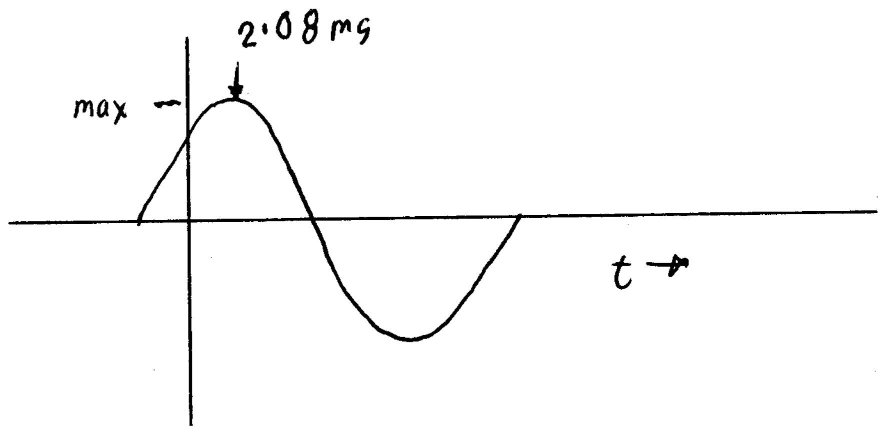
- to find the phase displacement:
- note for cos wave max occurs at

$$(\omega t + \alpha) = 0$$

Thus

$$\alpha = -\omega t = 377 x 2.08 x 10^{-3} = -0.784 rads = -\frac{\pi}{4}$$

$$= -0.784 x \frac{360}{2\pi} = -45 \; deg.$$



Periodic Waveforms

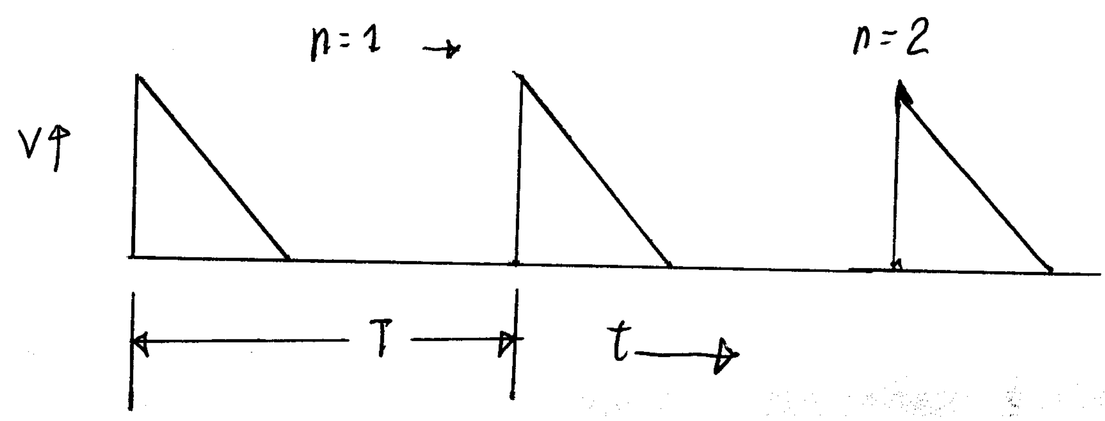
- Any shape of wave is periodic if it has a function

$$f(t) = f(t + nT)$$

where

t = time in sec
T = period = 1/f
n = any integer

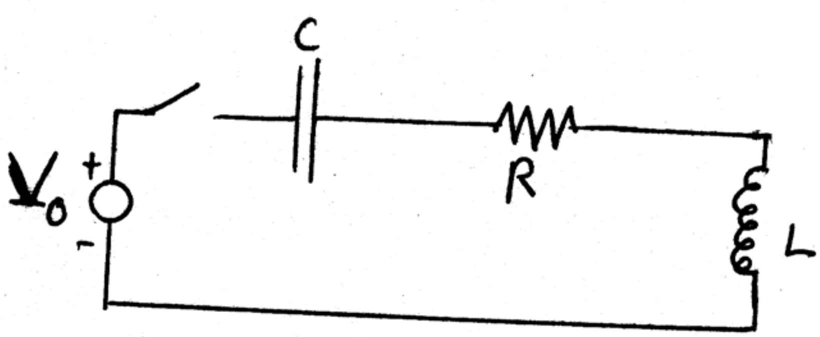
- NOTE: it can be shown that any periodic wave: can be made up of a series of sinusoidal waves

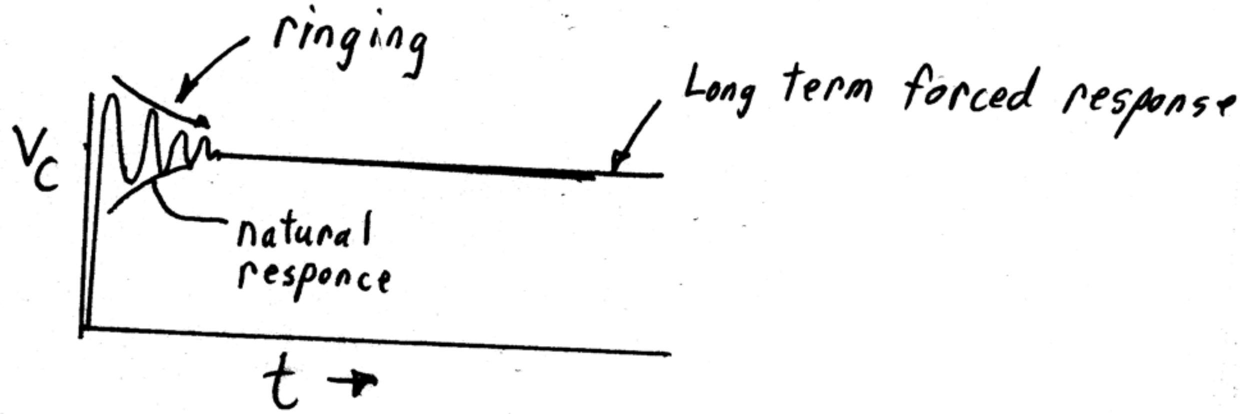


Complete Response

- Combines both natural and forced response
- Complete response: what happens to a sudden change
- e.g. Suddenly close a switch
- response is:

$$V_{complete} = V_{natural} + V_{forced}$$





Initial Underdamped Second Order Systems Con'd

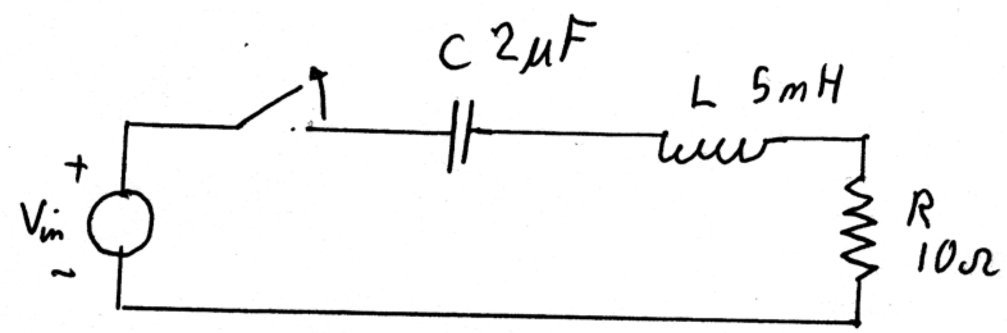
- for the example case L = 5 mH, C = 2 μF, R = 10 ohms
- Exposed to a square wave: 0 to 1 V changes.
- as solved in the RCL underdamped example

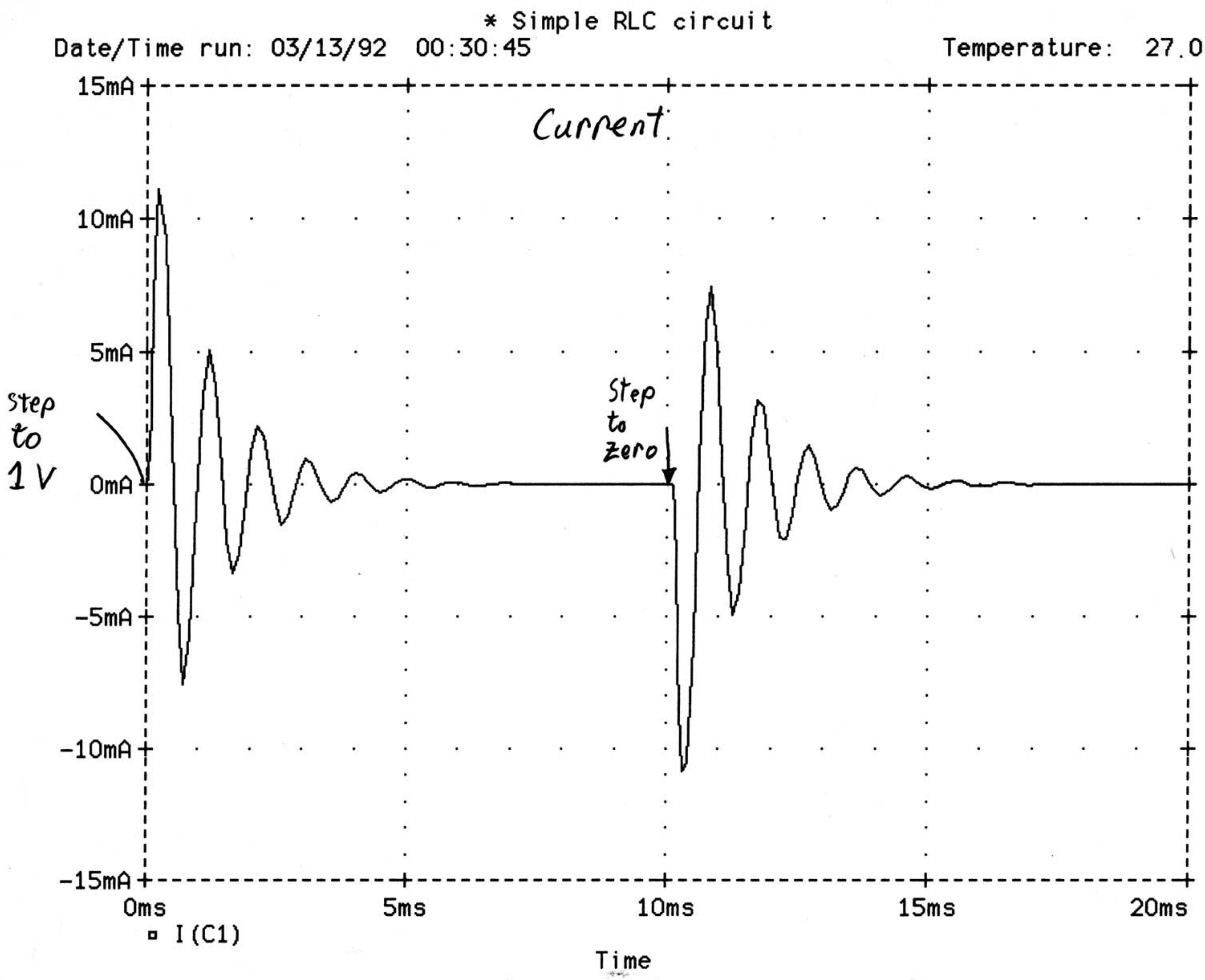
$$i(t) = A_1 \exp(-\alpha t) 2j \sin(\omega t)$$

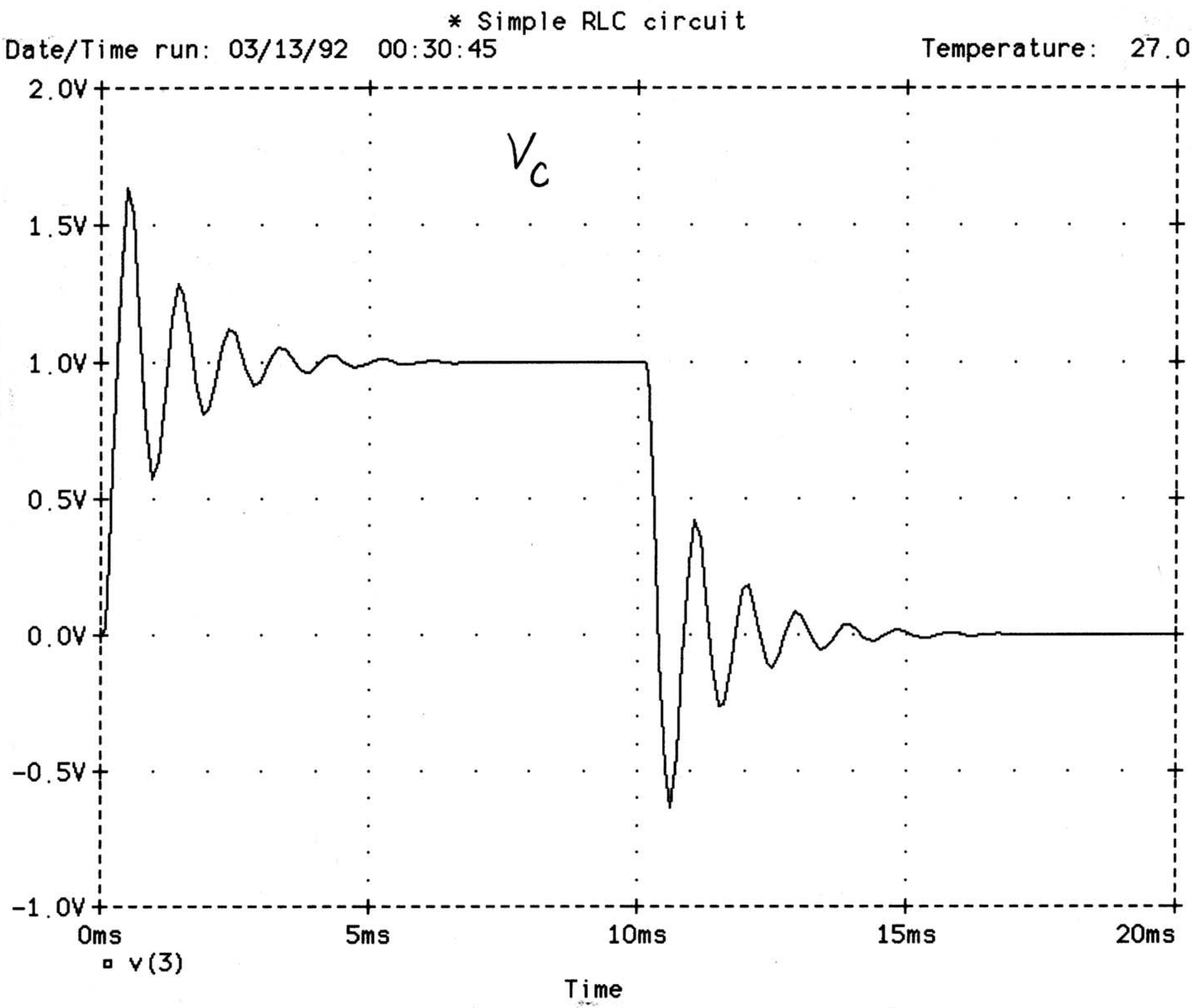
$$A_1 = \frac{V_c}{2j\omega L} = \frac{1}{2jx9.95x10^3x0.005} = \frac{20}{2j}\ mA$$

The 2j term is eliminates that from the sin function

$$i(t) = 20\exp(-10^3 t)\sin(9.95x10^3 t)\ mA$$




* Simple RLC circuit
Date/Time run: 03/13/92 00:30:45
Temperature: 27.0
Current
15mA
10mA
5mA
0mA
-5mA
-10mA
-15mA
step to 1V
Step to zero
0ms
5ms
10ms
15ms
20ms
I(C1)
Time


* Simple RLC circuit
Date/Time run: 03/13/92 00:30:45
Temperature: 27.0
Vc
2.0V
1.5V
1.0V
0.5V
0.0V
-0.5V
-1.0V
0ms
5ms
10ms
15ms
20ms
v(3)
Time

Complex numbers

- Imaginary number j

$$j = \sqrt{-1}$$

- Note: in math imaginary number is called i
- complex numbers involve real and imaginary parts

$$\vec{W} = Real(W) + j\ Imaginary(W)$$

Example:

$$\vec{W} = 1 + j2$$

$$Real(W) = R_W = 1 \qquad Imaginary(W) = I_W = 2$$

- often put as

$$\vec{W} = R_W + jI_W$$

Complex Numbers Plotted

- In electronics plot on X-Y axis
- X axis real, Y axis is imaginary
- A vector represents the imaginary number has length
- Vector has a magnitude M
- Vector is at some angle theta to the real axis
- Then the real and imaginary parts are

$$Real(W) = R_W = M\cos(\theta)$$

$$Imaginary(W) = I_W = M\sin(\theta)$$

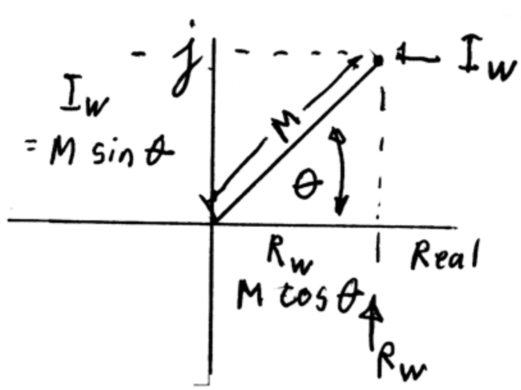
$$\vec{W} = M[\cos(\theta) + j\sin(\theta)]$$

- The magnitude

$$Magnitude(W) = (R_W^2 + I_W^2)^{1/2}$$

- And the angle is

$$\theta = arctan\left(\frac{I_W}{R_W}\right)$$



Complex Numbers and Exponentials

- What do complex numbers mean inside an exponential?
- Euler's Formula defines

$$\exp(j\theta) = \cos(\theta) + j\sin(\theta)$$

- thus

$$\cos(\theta) = \frac{1}{2}[\exp(j\theta) + \exp(-j\theta)]$$

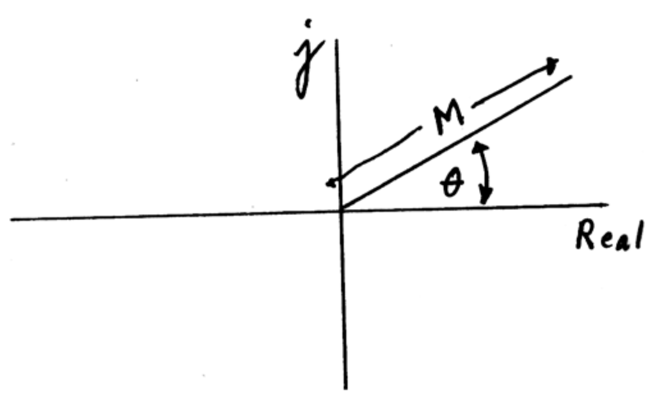
$$\sin(\theta) = \frac{1}{2j}[\exp(j\theta) - \exp(-j\theta)]$$

- Thus can represent a complex number as:

$$\vec{W} = M\exp(j\theta)$$

- called the exponential or polar form.
- can be shown as

$$\vec{W} = M \angle\underline{\theta}$$



Polar Coordinate System

- Rectangular (Cartesian or X-Y) Coordinates:
- Coordinates in terms of X and Y projection
- Polar Coordinates: specify in radius R, & angle θ
- Translation from Polar to Rectangular

$$x = R\cos\theta$$

$$y = R\sin\theta$$

- Translation form Rectangular to Polar

$$|R| = \left[x^2 + y^2\right]^{0.5}$$

$$\theta = arctan\left[\frac{y}{x}\right]$$

- Many Calculators have Rectangular to Polar conversion
- Use those for fastest complex operations
- Some also have special complex operations

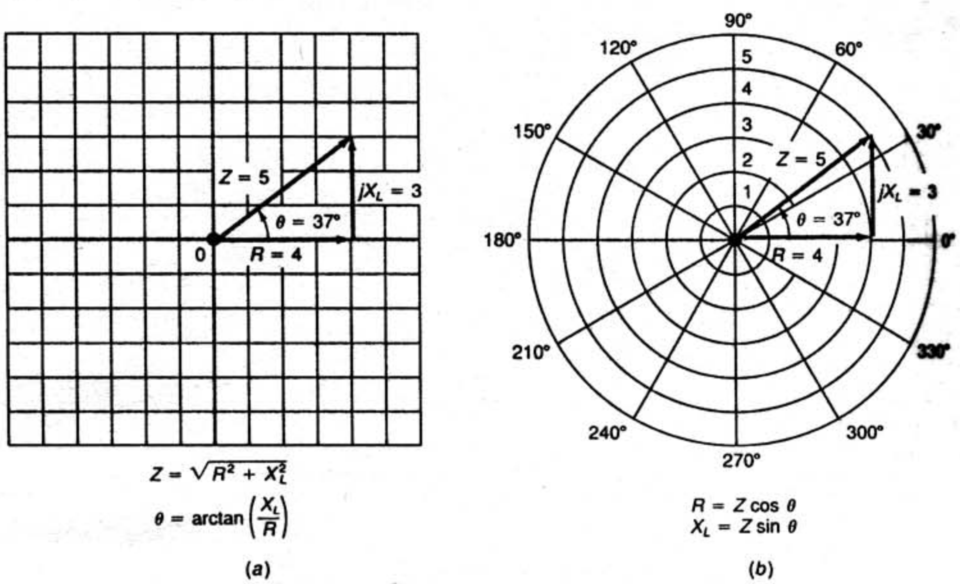


Fig. 25-8 Magnitude and angle of a complex number. (*a*) Rectangular form. (*b*) Polar form.

Complex Arithmetic

- Complex numbers behave like vectors.
- Adding or subtracting:

complex and real added (subtracted) separately

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

- Multiplication of complex numbers easiest with Polar

$$R_1\exp(-j\theta_1)R_2\exp(-j\theta_2) = R_1R_2\exp(-j[\theta_1 + \theta_2])$$

- Division using the real and imaginary parts

$$(a + jb)(c + jd) = (ac - db) + j(ad + bc)$$

- Division of complex numbers much easier with Polar

$$\frac{R_1\exp(-j\theta_1)}{R_2\exp(-j\theta_2)} = \frac{R_1}{R_2}\exp(-j[\theta_1 - \theta_2])$$

- Division using the real and imaginary parts
- for this bring the divisor to a real number

$$\frac{(a + jb)}{(c + jd)} = \frac{(a + jb)}{(c + jd)}\frac{(c - jd)}{(c - jd)}$$

$$= \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

Example Complex Arithmetic

- Addition

$$A = 3 + 5j \qquad \text{and} \qquad B = 5 + 10j$$

$$A + B = (3 + 5) + (5 + 10)j = 8 + 15j$$

In polar form

$$\theta = arctan\left[\frac{10}{8}\right] = 61.9^o$$

$$|A + B| = \left[8^2 + 15^2\right]^{0.5} = 17$$

$$A + B = 17/\underline{61.9^o}$$

- Multiplication and Division

$$C = 3 + 4j = 5/\underline{53.1^o}$$

$$D = 24 + 7j = 25/\underline{16.3^o}$$

$$C \ x \ D = 5 \ x \ 25/\underline{53.1^o + 16.3^o} = 125/\underline{69.4^o} = 44 + 117j$$

$$\frac{C}{D} = \frac{25}{5}/\underline{16.3^o - 53.1^o} = 5/\underline{-36.8^o} = 4 - 3j$$

Rotating vectors or Phasors

- What if we have an exponential dependent on time

$$\vec{W} = M\exp[j(\omega t + \theta)]$$

where t = the time

- Consider this to vector, length M
- it is rotating about the complex axis
- its angular velocity of rotation is ωt

$$\exp(j[\omega t + \theta]) = \cos(\omega t + \theta) + j\sin(\omega t + \theta)$$

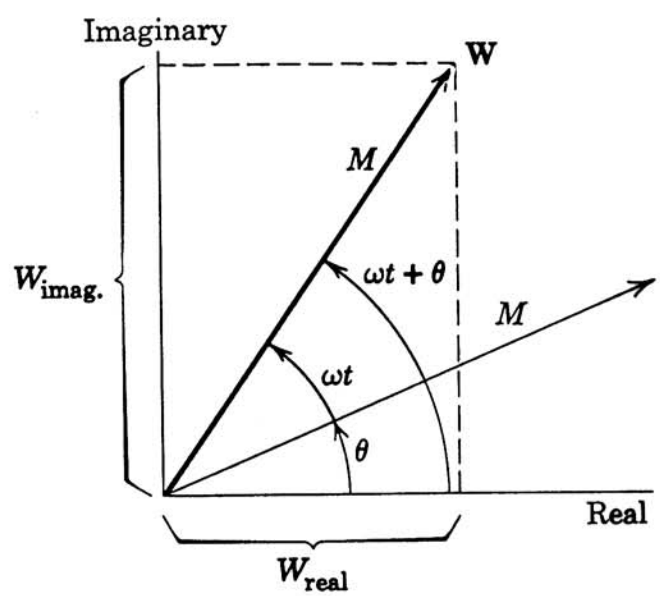
- This is the representation of a sine wave of frequency
- the real portion is the measured value of the wave t

$$Real[\exp(j[\omega t + \theta])] = \cos(\omega t + \theta)$$

- the phase factor is θ
- the imaginary portion:

where the phasor is at a given instance

$$Imaginary[\exp(j[\omega t + \theta])] = j\sin(\omega t + \theta)$$



Example Phasor formula

- What is the complex phasor representation for 60 Hz sine wave 12 V peak with a 45 degree phase delay and what is its value at t = 2.08 msec

- for the phase delay

$$\theta = \frac{\pi}{4}$$

- the angular frequency is:

$$\omega = 2\pi f = 2\pi 60 = 377$$

- thus the phasor representation is

$$V(t) = 12\exp\left(j\left[377t + \frac{\pi}{4}\right]\right)$$

Example Phasor formula Con'd

- at t = 2.08 msec value is

$$V(t{=}2.08) = 12\exp\left\{j\left[377x\,2.08 + \frac{\pi}{4}\right]\right\} = 12\exp\left\{j\left[\frac{\pi}{4} + \frac{\pi}{4}\right]\right\}$$

- the real value is:

$$\mathit{Real}\,[V(t{=}2.08)] = 12\cos(\frac{\pi}{2}) = 0$$

- the imaginary value (out of phase portion of wave)

$$\mathit{Imaginary}\,[V(t{=}2.08)] = 12\sin(\frac{\pi}{2}) = 12$$

- thus as expected this is the zero point of the wave