Parallel RLC Second Order Systems

- Consider a parallel RLC
- Switch at $t=0$ applies a current source
- For parallel will use KCL
- Proceeding just as for series but now in voltage

(1) Using KCL to write the equations:

$$C \frac{di}{dt} + \frac{v}{R} + \frac{1}{L} \int_0^t v \, dt = I_0$$

(2) Want full differential equation
- Differentiating with respect to time

$$C \frac{d^2v}{dt^2} + \frac{1}{R} \frac{dv}{dt} + \frac{1}{L} v = 0$$

(3) This is the differential equation of second order
- Second order equations involve 2nd order derivatives
Solving the Second Order Systems Parallel RLC

• Continuing with the simple parallel RLC circuit as with the series (4) Make the assumption that solutions are of the exponential form:

\[ i(t) = A \exp(st) \]

• Where \( A \) and \( s \) are constants of integration.
• Then substituting into the differential equation

\[
C \frac{d^2v}{dt^2} + \frac{1}{R} \frac{dv}{dt} + \frac{1}{L} v = 0
\]

\[
Cs^2 A \exp(st) + \frac{1}{R} sA \exp(st) + \frac{A}{L} \exp(st) = 0
\]

• Dividing out the exponential for the characteristic equation

\[
s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0
\]

• Giving the Homogeneous equation
• Get the 3 same types of solutions but now in voltage
• Just parameters are going to be different
General Solution Parallel RLC

- Solving the homogeneous quadratic as before
  \[ s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0 \]

- The general solution is:
  \[ s = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \]

- Note the difference from the series RLC
  \[ s_{\text{series}} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]

- Note the difference is in the damping term first term
- Again type of solution is set by the Descriminant
  \[ D = \left[ \left(\frac{1}{2RC}\right)^2 - \frac{1}{LC} \right] \]

- Recall RC is the time constant of the resistor capacitor circuit
3 solutions of the Parallel RLC

- What the Descriminant represents is about energy flows

\[ D = \left[ \left( \frac{1}{2RC} \right)^2 - \frac{1}{LC} \right] \]

- Again how fast is energy transferred from the L to the C
- How fast is energy lost to the resistor
- Get the same three cases & general equations set by D
- D > 0 : roots real and unequal: overdamped case
- D = 0 : roots real and equal: critically damped case
- D < 0 : roots complex and unequal: underdamped case
- Now the damping term changes

\[ \alpha_{\text{parallel}} = \frac{1}{2RC} \]

- For the series RLC it was

\[ \alpha_{\text{series}} = \frac{R}{2L} \]

- Recall \( \tau = RC \) for the resistor capacitor circuit
- While \( \tau = \frac{R}{L} \) for the resistor inductor circuit
- The natural frequency (underdamped) stays the same

\[ \omega_n = \frac{1}{\sqrt{LC}} \]

The difference is in the solutions created by the initial conditions
Forced Response & RL, RC and RLC Combination

- Natural Response: energy stored then decays
- Forced Response: voltage/current applied
- Forcing function can be anything
- Typical types are steps or sine functions
- Step response: called complete response in book
- Step involves both natural and forced response
- Forced response (Book): after steady state reached
- forced response: when forcing function applied.
- Forcing function: any applied V or I
- Most important case simple AC response
Forced Response

- How does a circuit act to a driving V or I which changes with time
- Assume this is long after the function is applied
- Problem easiest for RC & RL
- General problem difficult with RLC type
- Procedure: write the KVL or KCL laws
- Equate it to the forcing function \( F(t) \)

\[
F(t) = \sum_{j=1}^{n} v_j
\]

- Then create and solve Differential Equation

General solution difficult
Two simple Cases important:

(1) Steady V or I applied, or sudden changes at long intervals
- Just need to know how the C or L respond
- In long time C become open, L a short
- Solved as in RL and RC case
- Must have time between changes >> time constants

(2) Sinewave AC over long time
- Solved using the complex Impedance
Complete Response

• Complete response: what happens to a sudden change
• Apply a forcing function to the circuit (eg RC, RL, RLC)

• Complete response is a combination two responses

(1) First solve natural response equations
• use either differential equations
• Get the roots of the exp equations
• Or use complex impedance (coming up)

(2) Then find the long term forced response

(3) Add the two equations
\[ V_{\text{complete}} = V_{\text{natural}} + V_{\text{forced}} \]

(4) Solve for the initial conditions
Example Complete Response of RL to a step

- Consider an RL circuit with a switched voltage
- This is called a voltage step

(1) From previous results the natural response is:

\[ I(t)_{nat} = A \exp\left[-\frac{Rt}{L}\right] \]

(2) In the long term the L acts as an short

\[ I(t \to \infty)_{for} = \frac{V_{in}}{R} \]

(3) Combine the equations

\[ I(t) = \frac{V_{in}}{R} + A \exp\left[-\frac{Rt}{L}\right] \]
Complete Response of RL Con’d

(4) Solve for initial conditions
At t=0+ no current must flow

\[ 0 = I(0) = \frac{V_{in}}{R} + A \exp\left[-\frac{RO}{L}\right] \]

\[ A = -\frac{V_{in}}{R} \]

- thus the complete response equation is

\[ I(t) = \frac{V_{in}}{R} \left(1 - \exp\left[-\frac{Rt}{L}\right]\right) \]
Complete Response In General

- General solution difficult
- Two simple Cases important:

(1) Steady V or I applied
  - just need to know how the C or L respond
  - in long time C become open, L a short
  - solved as in RL and RC case

(2) Sinewave AC over long time
  - solved using the complex Impedance (EC 10)
Complete Response of Second Order Circuits

- 2nd order complete response procedure same as 1st
- but getting coefficients more complex

(1) First solve natural response equations
- use either differential equations
- or impedance method (next section)

(2) Then find the long term forced response
- for sudden DC changes get steady state results

(3) Add the two equations
\[ I_{\text{complete}} = I_{\text{natural}} + I_{\text{forced}} \]

(4) Solve for the initial conditions.
- must use initial conditions
- For current need both
  \[ I(t=0+) \quad \frac{dI(t=0+)}{dt} \]
- similar requirements for Voltage
Complete Response of series RLC to a Voltage step

- Consider an RLC circuit with a switched voltage
- Find the voltage across the capacitor

(1) From previous results for the natural response:
- in terms of the solution s of this impedance then
  \[ I(t)_{nat} = A_1 \exp(s_1 t) + A_2 \exp(s_2 t) \]

- where s' are the roots of the homogeneous equation

(2) In long term, L acts as an short, C open
  \[ I(t \to \infty)_{for} = 0 \]

(3) Combine the equations
  \[ I(t)_{nat} = A_1 \exp(s_1 t) + A_2 \exp(s_2 t) \]
Complete Response of RLC series Con’d

(4) Solve for initial conditions
At t=0+ no current must flow, L open

\[ 0 = I(0) = A_1 \exp(s_1 0) + A_2 \exp(s_2 0) \]

\[ A_1 = -A_2 \]

- Also need the derivative at t=0+
- noting that L is open and C uncharged then

\[ V_{in} = L \frac{di}{dt} \]

- thus:

\[ \frac{V_{in}}{L} = A (s_1 \exp(s_1 0) - s_2 \exp(s_2 0)) \]

\[ A = \frac{V_{in}}{L (s_1 - s_2)} \]

- the exact form depends on the type of roots.
Complete Response of RLC series Cont'd

- if roots are complex then
  \[ s_1 = -\alpha + j\omega \quad s_2 = -\alpha - j\omega \]

- and the coefficient is
  \[ A = \frac{V_{in}}{L(s_1 - s_2)} = \frac{V_{in}}{jL \, 2\omega} \]

- since
  \[ \sin(\omega t) = \frac{1}{2j} [\exp(j\omega t) - \exp(-j\omega t)] \]

  \[ I(t)_{nat} = \frac{V_{in}}{L \omega} \exp(-\alpha t) \sin(\omega t) \]

- How to solve complete response for AC sources

- Need the complex impedance methods

![RLC Circuit Diagram]
Sinusoidal waves

- Most circuits involve periodic signals
- Most common are made of sinusoidal waves

\[ a(t) = A_0 \cos(\omega t + \alpha) \]

Where
- \( a(t) \) = signal at time \( t \)
- \( A_0 \) = max signal or amplitude
- \( \omega \) = (omega) radial frequency, radians/sec.
- \( t \) = time in sec.
- \( \alpha \) = phase shift angle in radians (also \( \phi \), phi)

- radial frequency related to the regular frequency \( f \) by

\[ \omega = 2\pi f \]

- The period of the wave is

\[ T = \frac{1}{f} = \frac{2\pi}{\omega} \]
Sin waves

- can relate a cos wave to a sin wave via

\[ a(t) = A_0 \cos(\omega t + \alpha) = A_0 \sin(\omega t + \alpha + \frac{\pi}{2}) \]

- sine wave: phase shifted by 90 degrees from cos

In each graph \( x \) is in radians.

5.22 \[ y = \sin x \]

5.23 \[ y = \cos x \]
Projection of Cos wave

- Useful to regard wave as a rotating vector
- Rotating vector is called a phasor
- Period of rotation

\[ \text{period} = \frac{2\pi}{\omega} \]

- Instantaneous value = projection on x axis of vector

![Diagram of projections](image)

**Figure 3.8** Projections of a rotating line.

- Call electrical signals of this type "Alternating Current"
Example of sinusoidal wave

- Example: a 60 Hz wave has its maximum at 2.08 msec.
- find the wave formula

- first find the radial frequency
  \[ \omega = 2\pi f = 2\pi 60 = 377 \text{ rads/sec} \]

- to find the phase displacement:
- note for cos wave max occurs at
  \[ (\omega t + \alpha) = 0 \]

Thus

\[ \alpha = -\omega t = 377 \times 2.08 \times 10^{-3} = -0.784 \text{ rads} = -\frac{\pi}{4} \]

\[ = -0.784 \times \frac{360}{2\pi} = -45 \text{ deg.} \]
Periodic Waveforms

- Any shape of wave is periodic if it has a function
  \[ f(t) = f(t + nT) \]

where
- \( t \) = time in sec
- \( T \) = period = \( 1/f \)
- \( n \) = any integer

- NOTE: it can be shown that any periodic wave: can be made up of a series of sinusoidal waves
Complete Response

- Combines both natural and forced response
- Complete response: what happens to a sudden change
- e.g. Suddenly close a switch
- response is:

\[ V_{\text{complete}} = V_{\text{natural}} + V_{\text{forced}} \]
Initial Underdamped Second Order Systems Con'd

- for the example case $L = 5 \, \text{mH}$, $C = 2 \, \mu\text{F}$, $R = 10 \, \text{ohms}$
- Exposed to a square wave: 0 to 1 V changes.
- as solved in the RCL underdamped example

$$i(t) = A_1 \exp(-\alpha t) 2j \sin(\omega t)$$

$$A_1 = \frac{V_c}{2j \omega L} = \frac{1}{2j \times 9.95 \times 10^3 \times 0.005} = \frac{20}{2j} \, \text{mA}$$

The $2j$ term is eliminated from the sin function

$$i(t) = 20 \exp(-10^3 t) \sin(9.95 \times 10^3 t) \, \text{mA}$$
* Simple RLC circuit

Date/Time run: 03/13/92 00:30:45  
Temperature: 27.0

Current:

Step to
1V

Step to
Zero

I(C1)

Time
Complex numbers

• Imaginary number j
  \[ j = \sqrt{-1} \]

• Note: in math imaginary number is called i
• complex numbers involve real and imaginary parts
  \[ \vec{W} = \text{Real}(W) + j \text{Imaginary}(W) \]

Example:

\[ \vec{W} = 1 + j2 \]

\[ \text{Real}(W) = R_W = 1 \quad \text{Imaginary}(W) = I_W = 2 \]

• often put as
  \[ \vec{W} = R_W + jI_W \]
Complex Numbers Plotted

- In electronics plot on X-Y axis
- X axis real, Y axis is imaginary
- A vector represents the imaginary number has length
- Vector has a magnitude $M$
- Vector is at some angle theta to the real axis
- Then the real and imaginary parts are

  $\text{Real} (W) = R_W = M \cos(\theta)$

  $\text{Imaginary} (W) = I_W = M \sin(\theta)$

  $\vec{W} = M [\cos(\theta) + j \sin(\theta)]$

- The magnitude

  $\text{Magnitude} (W) = (R_W^2 + I_W^2)^{1/2}$

- And the angle is

  $\theta = \arctan \left( \frac{I_W}{R_W} \right)$
Complex Numbers and Exponentials

- What do complex numbers mean inside an exponential?
- Euler’s Formula defines

\[ \exp(j \theta) = \cos(\theta) + j \sin(\theta) \]

- thus

\[ \cos(\theta) = \frac{1}{2} [\exp(j \theta) + \exp(-j \theta)] \]

\[ \sin(\theta) = \frac{1}{2j} [\exp(j \theta) - \exp(-j \theta)] \]

- Thus can represent a complex number as:

\[ \vec{W} = M \exp(j \theta) \]

- called the exponential or polar form.
- can be shown as

\[ \vec{W} = M \ / \theta \]
Polar Coordinate System

- Rectangular (Cartesian or X-Y) Coordinates:
- Coordinates in terms of X and Y projection
- Polar Coordinates: specify in radius R, & angle θ
- Translation from Polar to Rectangular
  \[ x = R \cos \theta \]
  \[ y = R \sin \theta \]

- Translation from Rectangular to Polar
  \[ |R| = \left[ x^2 + y^2 \right]^{0.5} \]
  \[ \theta = \arctan \left( \frac{y}{x} \right) \]

- Many Calculators have Rectangular to Polar conversion
- Use those for fastest complex operations
- Some also have special complex operations

Fig. 25-8 Magnitude and angle of a complex number. (a) Rectangular form. (b) Polar form.
Complex Arithmetic

- Complex numbers behave like vectors.
- Adding or subtracting:
  complex and real added (subtracted) separately
  \[(a + jb) + (c + jd) = (a + c) + j(b + d)\]
  \[(a + jb) - (c + jd) = (a - c) + j(b - d)\]

- Multiplication of complex numbers easiest with Polar
  \[R_1 \exp(-j\theta_1)R_2 \exp(-j\theta_2) = R_1R_2 \exp(-j[\theta_1 + \theta_2])\]

- Division using the real and imaginary parts
  \[(a + jb)(c + jd) = (ac - db) + j(ad + bc)\]

- Division of complex numbers much easier with Polar
  \[
  \frac{R_1 \exp(-j\theta_1)}{R_2 \exp(-j\theta_2)} = \frac{R_1}{R_2} \exp(-j[\theta_1 - \theta_2])
  \]

- Division using the real and imaginary parts
- for this bring the divisor to a real number
  \[
  \frac{(a + jb)}{(c + jd)} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}
  \]
Example Complex Arithmetic

• Addition

\[ A = 3 + 5j \quad \text{and} \quad B = 5 + 10j \]

\[ A + B = (3 + 5) + (5 + 10)j = 8 + 15j \]

In polar form

\[ \theta = \arctan \left( \frac{10}{8} \right) = 61.9^\circ \]

\[ |A + B| = \left[ 8^2 + 15^2 \right]^{0.5} = 17 \]

\[ A + B = \frac{17}{61.9^\circ} \]

• Multiplication and Division

\[ C = 3 + 4j = \frac{5}{53.1^\circ} \]

\[ D = 24 + 7j = \frac{25}{16.3^\circ} \]

\[ C \times D = 5 \times \frac{25}{53.1^\circ} + 16.3^\circ = \frac{125}{69.4^\circ} = 44 + 117j \]

\[ \frac{C}{D} = \frac{25/16.3^\circ - 53.1^\circ}{5} = \frac{5}{-36.8^\circ} = 4 - 3j \]
Rotating vectors or Phasors

- What if we have an exponential dependent on time
  \[ \mathbf{W} = M \exp[j(\omega t + \theta)] \]

  where \( t \) = the time

- Consider this to vector, length \( M \)
- it is rotating about the complex axis
- its angular velocity of rotation is \( \omega t \)
  \[ \exp(j[\omega t + \theta]) = \cos(\omega t + \theta) + j\sin(\omega t + \theta) \]

- This is the representation of a sine wave of frequency
- the real portion is the measured value of the wave \( t \)
  \[ \text{Real}[\exp(j[\omega t + \theta])] = \cos(\omega t + \theta) \]

- the phase factor is \( \theta \)
- the imaginary portion:
  where the phasor is at a given instance
  \[ \text{Imaginary}[\exp(j[\omega t + \theta])] = j\sin(\omega t + \theta) \]
Example Phasor formula

- What is the complex phasor representation for 60 Hz sine wave 12 V peak with a 45 degree phase delay and what is its value at t = 2.08 msec

- for the phase delay
  \[ \theta = \frac{\pi}{4} \]

- the angular frequency is:
  \[ \omega = 2\pi f = 2\pi 60 = 377 \]

- thus the phasor representation is
  \[ V(t) = 12\exp\left(j \left[ 377t + \frac{\pi}{4} \right]\right) \]
Example Phasor formula Con'd

- at \( t = 2.08 \) msec value is

\[
V(t=2.08) = 12\exp\left(j\left[377 \times 2.08 + \frac{\pi}{4}\right]\right) = 12\exp\left(j\left[\frac{\pi}{4} + \frac{\pi}{4}\right]\right)
\]

- the real value is:

\[
Real[V(t=2.08)] = 12\cos\left(\frac{\pi}{2}\right) = 0
\]

- the imaginary value (out of phase portion of wave)

\[
Imaginary[V(t=2.08)] = 12\sin\left(\frac{\pi}{2}\right) = 12
\]

- thus as expected this is the zero point of the wave