Quality factor $Q$ and Filters

- Changing $R$ changes the resonance width
- Controlled by the energy loss per cycle
- Called the damping factor
- Define the Quality Factor $Q$ as

$$Q = 2\pi \frac{\text{max energy stored}}{\text{energy lost per cycle}}$$

- The $Q$ measures how good a circuit is
- The higher the $Q$, the sharper the peak
Quality factor $Q$ in Filters

- Energy loss is in the resistor

$$W_{cycle} = \frac{I_{rms}^2 R}{f_0} = \frac{2\pi I_{rms}^2 R}{\omega_0}$$

- Max energy stored in the Inductor is

$$W_{L_{max}} = \frac{i^2L}{2} = I_{rms}^2 L$$

- Thus the $Q$ factor is

$$Q = 2\pi \frac{I_{rms}^2 L}{2\pi I_{rms}^2 R \frac{\omega_0}{R}} = \frac{\omega_0 L}{R}$$

- Similarly for the capacitor

$$Q = \frac{1}{\omega_0 CR}$$
Normalized Filter Response (EC 14.2)

- Filter equations can be expressed in $Q$
- Consider the series RLC circuit

$$Z = \frac{1}{Y} = R + j\left[\omega L - \frac{1}{\omega C}\right]$$

- At the resonance frequency

$$Z_0 = R = \frac{1}{Y_0}$$

- Relative to the resonance values

$$\frac{Y}{Y_0} = \frac{Z_0}{Z} = \frac{1}{1 + j\left[\frac{\omega L}{R} - \frac{1}{\omega CR}\right]}$$

- Called the Normalized Response
Quality factor $Q$ & Filters

- Since the Quality Factor is
  
  $$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$
  
- thus
  
  $$\frac{L}{R} = \frac{Q}{\omega_0}$$

  $$\frac{1}{CR} = Q \omega_0$$

- Thus can write the Normalized Response
  
  $$\frac{Y}{Y_0} = \frac{Z_0}{Z} = \frac{1}{1 + jQ \left[ \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right]}$$

- this same equation works for parallel RLC circuit or any other simple resonance circuit
- only $Q$'s and $\omega_0$ change
Bandwidth and Q in Filters

- Want to measure the Bandwidth
- Bandwidth: frequency range between the 70% points
- in the form

\[
\frac{Y}{Y_0} = \frac{Z_0}{Z} = \frac{1}{1 + jQ \left[ \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right]}
\]

- thus want the point where

\[
\frac{Y}{Y_0} = \frac{1}{1 \pm j1}
\]

- thus

\[
\left[ \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right] = \pm \frac{1}{Q}
\]

Figure 7.22 The normalized response of a series RLC circuit.
Bandwidth and Q in Filters

- Define 2 half power or 70% points frequencies
- $\omega_1$ = lower frequency

$$\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1} = -\frac{1}{Q}$$

- and for $\omega_2$ = upper frequency

$$\frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2} = +\frac{1}{Q}$$
• Solving for these at the lower frequency

\[
\frac{\omega_1^2 - \omega_0^2}{\omega_1 \omega_0} = -\frac{1}{Q}
\]

• solving the quadratic gives

\[
\omega_1 = \omega_0 \left[ 1 + \left( \frac{1}{2Q} \right)^2 \right]^{1/2} \frac{\omega_0}{2Q}
\]

• similarly for the upper frequency

\[
\omega_2 = \omega_0 \left[ 1 + \left( \frac{1}{2Q} \right)^2 \right]^{1/2} + \frac{\omega_0}{2Q}
\]

• Define bandwidth from the 70% points as

\[
\text{Bandwidth} = \omega_2 - \omega_1 = \frac{\omega_0}{Q}
\]

\[
\text{Bandwidth} = f_2 - f_1 = \frac{f_0}{Q}
\]
Bandwidth and Q in Filters

- thus bandwidth determined by the Quality Factor

- One approximation useful
  - When \( Q \geq 10 \) then with \(< 2\% \) error
  \[
  \left[ 1 + \left( \frac{1}{2Q} \right)^2 \right]^{1/2} \approx 1
  \]

- Thus can approximate
  \[
  \omega_1 \approx \omega_0 \left[ 1 - \frac{1}{2Q} \right] \quad \omega_2 \approx \omega_0 \left[ 1 + \frac{1}{2Q} \right]
  \]

- and the curve is symmetric about \( \omega_0 \)
  \[
  \omega_2 - \omega_0 = \omega_0 - \omega_1 = \frac{\omega_0}{2} Q
  \]
Example Bandwidth and Q in Filters

- Design a parallel RLC circuit to select the 1000 KHz frequency of AM radio, with a bandwidth of 5 KHz. Find the C & R needed for the circuit when 
  - L = 20 µH

- Since for the parallel RLC

\[ Y = G + j \left[ \omega C - \frac{1}{\omega L} \right] \]

- thus

\[ \omega_0 = \frac{1}{\sqrt{L C}} \]

\[ C = \frac{1}{\omega_0^2 L} = \frac{1}{(2\pi 10^6)^2 \times 2 \times 10^{-5}} = 1.27 \times 10^{-9} = 1.27 \text{ nF} \]
Example Bandwidth and $Q$ in Filters Cont’d

• to get the desired resistance note

$$Bandwidth = f_2 - f_1 = \frac{f_0}{Q}$$

$$Q = \frac{f_0}{f_2 - f_1} = \frac{10^6}{5 \times 10^3} = 200$$

• Since for the parallel RLC

$$Q = \frac{1}{\omega_0 L G}$$

$$R = Q \omega_0 L = 200 \times 6.14 \times 10^6 \times 2 \times 10^{-5} = 25.1 \text{ Kohms}$$
Phasors: Peak and RMS values (EC 10)

- NOTE: with phasors there are two used values
- work with Peak V or I (best for where plotting is needed)
- work with RMS V or I (best for power calculations)
- either are correct
- Do not mix the two types in one problem

- example

\[ V = 10 \cos(\omega t + 45^\circ) \]

- In peak value phasor form:

\[ \vec{V} = 10/45^\circ \]

- In RMS value phasor form:

\[ \vec{V} = \frac{10}{\sqrt{2}}/10^\circ = 7.07/45^\circ \]
Power Calculation in AC

- Recall Power is always given by
  \[ P = I^2 Z = \frac{V^2}{Z} \]

- NOTE: for this must use RMS phasors
- For AC waves the L and C produce only imaginary
- For Inductor

  \[ i(t) = I_0 \cos(\omega t) = \frac{I_0}{\sqrt{2}} \cdot 0 \]

  \[ V_L(t) = I j \omega L = \frac{I_0 \omega L}{\sqrt{2}} / 90^\circ \]

  \[ v_L(t) = \frac{I_0 \omega L}{\sqrt{2}} \sin(\omega t) - \pi \]

- thus the instantaneous power is

  \[ P = i(t)v(t) = \frac{2I_0^2 \omega L}{2} \sin(\omega t) \cos(\omega t) \]

  where

  \[ \sin(\omega t) \cos(\omega t) = 2\sin(2\omega t) \]
Reactive Power Calculation in AC

- since a sin wave averaged over two periods is
  \[ P_{\text{avg}} = \frac{1}{2} \int_0^T 2I_0^2 \omega L \sin(\omega t) \cos(\omega t) \, dt = 0 \]

- Thus the average power is zero

- However the reactive power stored is
  \[ P_{\text{peak}} = \frac{I_0^2 \omega L}{2} = I_{\text{rms}}^2 X \]

Where
\[ I_{\text{rms}} = \frac{I}{\sqrt{2}} \quad X = \omega L = \text{reactive impedance} \]

- Reactive power must be supplied
- but is returned over the cycle
AC Power Factor

- Real AC Power is affected by the V phase angle

\[ P(t) = V_{rms} \sqrt{2} \cos(\omega t + \theta) I_{rms} \sqrt{2} \cos(\omega t) \]

\[ = 2V_{rms} I_{rms} \cos(\omega t + \theta) \cos(\omega t) \]

\[ = V_{rms} I_{rms} \cos(\theta) + V_{rms} I_{rms} \cos(\omega t + \theta) \]

because

\[ 2\cos(A) \cos(B) = \cos(A - B) \cos(A + B) \]

- Since the average over time

\[ \frac{1}{T} \int_{0}^{T} V_{rms} I_{rms} \cos(\omega t + \theta) dt = 0 \]

- Thus the time averaged power is

\[ P_{avg} = V_{rms} I_{rms} \cos(\theta) \]

(a) Circuit diagram  (b) Phasor diagram  (c) Time variation
AC Power Factor

• The apparent power is in Volt-Amperes
  \[ P_{\text{apparent}} = V_{\text{rms}} I_{\text{rms}} \]

• VI related to the real power by the Power Factor pf
  \[ pf = \cos(\theta) = \frac{P_{\text{avg}}}{V_{\text{rms}} I_{\text{rms}}} \]

• NOTE: the power angle is that of the impedance
  \[ \theta_{P_{\text{avg}}} = \theta_{Z} \]

• Inductive power (most common): Lagging power factor
  I lags V in time, but Z angle is positive

• Capacitive Power: Leading power factor
  I leads V in time: but Z angle is negative
AC Reactive Power

- Reactive Power: Power must be supplied but will be returned in part of the cycle
  \[ P_X = Q = I_{rms}^2 X = I_{rms}^2 Z \sin(\theta) = V_{rms} I_{rms} \sin(\theta) \]

- units: Volts-Amperes Reactive (VAR)

- Power just like other complex unites
  \[ \vec{P}_{app} = S = P_R + jQ = |V_{rms} I_{rms}| / \theta \]

- Often draw the Power Triangle:
  \( I^2X \) plotted against \( I^2R \)
Example AC Power

- Example: AC electric motor has $R = 4$ ohms coils
  Inductance of 10 mH, what is power factor at 60 Hz, 120 V

- Impedance is
  
  $$Z = R + j\omega L = 4 + j377 \times 0.01 = 4 + j3.77 = 5.50/43.3^\circ$$

  $$\theta = \arctan\left(\frac{3.77}{4}\right) = 43.3^\circ$$

- Power factor is
  
  $$pf = \cos(43.3^\circ) = 0.728$$

- Current required is
  
  $$I_{rms} = \frac{V_{rms}}{Z} = \frac{120}{5.50/43.3^\circ} = 21.8/-43.3^\circ$$ Amps

- NOTE: this is the I that a current meter would read
Example AC Power continued

- Thus for AC power: Volt-Amps is:
  \[ VA = V_{\text{rms}} I_{\text{rms}} = 120 \times 21.8 = 2618 \text{ VA} \]

- Average Power consumed is
  \[ P_{\text{avg}} = V_{\text{rms}} I_{\text{rms}} pf = 2618 \times 0.729 = 1.905 \text{ KW} \]

**KW = KiloWatts**

\[ \overrightarrow{P_{\text{avg}}} = 1.905/43.3^\circ \]

- Reactive power is
  \[ Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta) \]
  \[ = 2618 \times \sin(43.3^\circ) = 2618 \times 0.689 = 1795 \text{ VAR} \]
Power Factor compensation

- Most heavy devices (motors, heater coils) inductive
- thus tendency is to get lagging power factor
- But large users charged extra for low power factor

reason:
- generators must supply the I(t): thus larger generators
- also power lines mostly resistive:
Thus VARS create higher real losses in the delivery lines
That loss is the Utilities cost, not the users!

- thus users must often build capacitor banks to
minimize their power factor charges
Power Factor Compensation cont’d

• To compensate want only real power

\[ 0 = X_L + X_C = j\omega L - j\frac{1}{\omega C} \]

• Thus

\[ C = \frac{1}{\omega^2 L} \]

• actually often rate the capacitor in its reactive VAR’s
• rated for specific frequency
• thus compensation is generally not perfect
• average power is the only thing that does real work
• thus compensation increases work that can be done
Example compensating for power factor

- What C compensation must be supplied for the motor of the previous example: \( R = 4 \) ohms coils
  Inductance of 10 mH, what is power factor at 60 Hz, 120 V

- Impedance is
  \[
  Z = R + j\omega L = 4 + j3.77 = 5.50/43.3^\circ
  \]

- Power factor is
  \[
  pf = \cos(43.3^\circ) = 0.728
  \]
  \[
  P_{avg} = V_{rms}I_{rms}pf = 2618 \times 0.729 = 1.905 \text{ KW}
  \]

- To compensate
  \[
  C = \frac{1}{\omega^2 L} = \frac{1}{377^2 \times 0.01} = \frac{1}{1421} = 704 \text{ } \mu F
  \]

- Note that adding C reduces the Z of device
- This makes more current available at given V
- Would increase the real power
- Thus increases the real work that can be done
Example compensating for power factor

- In practice would reduce the applied V
- and keep work done constant.
- eg. What C compensation must be supplied for the motor
- In terms of VAR’s need

\[ Q = V_{rms} I_{rms} \sin(\theta) = 2618 \times \sin(43.3^\circ) \]

\[ = 2618 \times 0.689 = -1.795 \text{ KVAR} \]

- Typical commercial unit 1.5 Kvar thus compensation

\[ Q_{comp} = 1795 - 1500 = 295 \text{ var} \]

- if keep work done by motor constant then
  real power remains constant. Thus as before

\[ P_{\text{avg:comp}} = P_{\text{avg:uncomp}} = V_{rms} I_{rms} p_f_{\text{uncomp}} \]

\[ = 2618 \times 0.729 = 1.905 \text{ KW} \]

\[ \theta_{\text{comp}} = \arctan \left( \frac{Q_{\text{comp}}}{P_{\text{avg}}} \right) = \arctan \left( \frac{295}{1905} \right) = \arctan(0.154) = 8.77^\circ \]

- thus the new power factor is

\[ p_f_{\text{comp}} = \cos(\theta_{\text{comp}}) = \cos(8.77^\circ) = 0.988 \]
Problem with power meters

- Getting real power meters is difficult

- older moving coil meters average VI
two coils let IV interact
Mechanical effects of needle average signal
Problem with frequency effects

- new electronic meters:
- "True RMS" measure IV instantaneously
convert to digital, do numerical integral
and time average