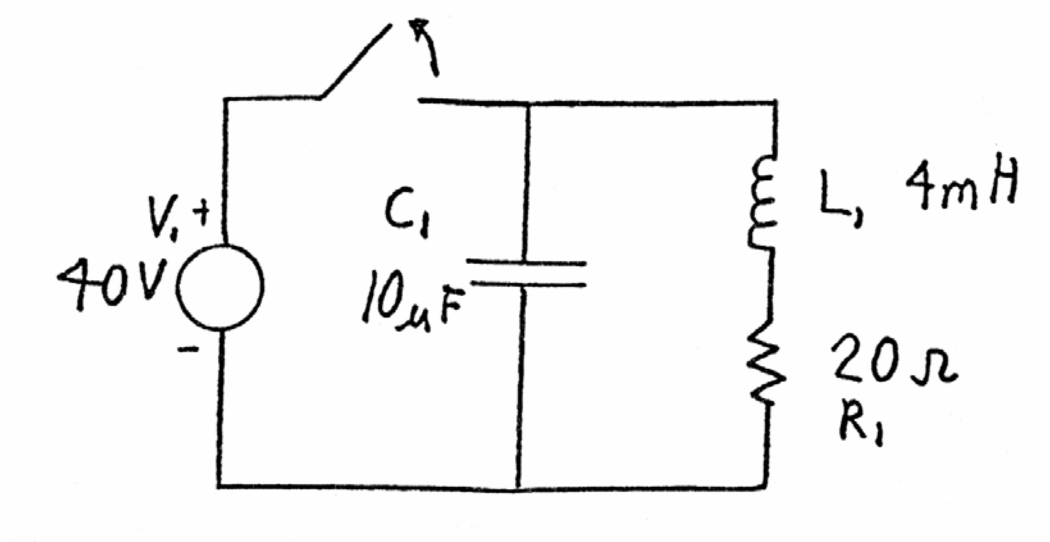
RLC Exam like Example

Consider this circuit where the switch has been closed long enough for the circuit to stabilized. Then open switch at t=0.



(a) Write the DE for circuit & initial conditions (current or voltage across each element).
(b) Solve DE for I through R_1
(c) What type of damping does this have? What R_1 is needed for the other damping types?
(d) For the (c) which is underdamped what is the 3 natural freq., damped freq. and damping factor?

(a) Using KVL

$$L\frac{di}{dt} + iR + \frac{1}{C}\int i \cdot dt = 0$$

- Differentiating

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} = 0$$

RLC Example Cont'd

- Initial conditions, C is fully charged, acts a V source

C has $V_C = 40V$

- L acts as a short before switch opens (no change in I)
- Therefore, current controlled by R

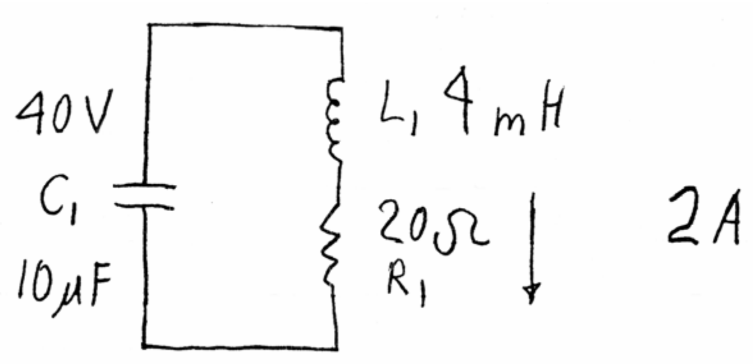
$$I_L = I_R = \frac{V}{R} = \frac{40}{20} = 2A$$

$$V_L = 0\,,\; V_R = IR = 40V$$

(b) Assuming an exponential solution then get the equations

$$I(t) = Ae^{st}$$

$$0 = s^2 L + sR + \frac{1}{C}$$



RLC Example Cont'd

- Solving this quadratic

$$0 = s^2 + s\frac{R}{L} + \frac{1}{CL}$$

- The roots are

$$s = \frac{R}{2L} \pm \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right]^{\frac{1}{2}}$$

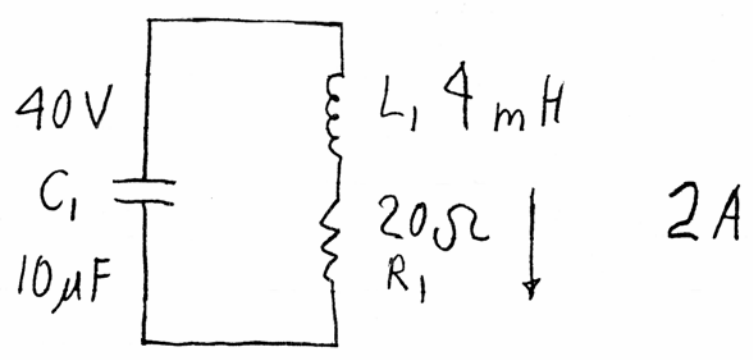
$$\frac{R}{2L} = \frac{20}{2\times 4\times 10^{-3}} = 2.5\times 10^{3}\ sec^{-1}$$

$$\frac{1}{LC} = \frac{1}{4\times 10^{-3}\times 10^{-5}} = 2.5\times 10^{7}\ sec^{-2}$$

- Thus the discriminate

$$\left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right] = \left[\left(2.5\times 10^{3}\right)^2 - 2.5\times 10^{7}\right] = -1.875\times 10^{7}\ sec^{-2}$$

- Because the discriminate is negative.
- This is an underdamped solution



RLC Example Cont'd

- Underdamped solved using 'α, ω' form

$$\alpha = \frac{R}{2L} = 2.5 \times 10^{3}\ sec^{-1}$$

$$\omega_n^2 = \frac{1}{LC} = 2.5 \times 10^{7} = 5000\ rad / sec$$

$$\therefore \omega^2 = \omega_n^2 - \alpha^2 = 2.5 \times 10^{7} - 6.25 \times 10^{6} = 1.875 \times 10^{7}$$

$$\omega = 4.33 \times 10^{3}$$

- Thus the solution will be of the form

$$i(t) = e^{-\alpha t}\left[A_1 e^{j\omega t} + A_2 e^{-j\omega t}\right]$$

- At t = 0, I = 2A

$$i(0) = e^{0}\left[A_1 e^{0} + A_2 e^{0}\right] = A_1 + A_2$$

$$A_1 + A_2 = 2A$$

RLC Example Cont'd

- As 2nd order also need the derivative at t = 0

$$L\frac{di}{dt} + I_0 R = V_C$$

$$L\frac{di}{dt} = V_C - I_0 R = 0$$

$$\frac{di}{dt} = 0$$

$$\therefore \frac{di(0)}{dt} = -\alpha e^{-\alpha t}\left[A_1 e^{j\omega t} + A_2 e^{-j\omega t}\right] + e^{\alpha t}\left[j\omega A_1 e^{j\omega t} - j\omega e^{-j\omega t}\right]$$

$$= -\alpha[A_1 + A_2] + j\omega[A_1 - A_2] = 0$$

$$\therefore j\omega[A_1 - A_2] = -\alpha[A_1 + A_2]$$

$$A_1 - A_2 = \frac{-\alpha}{j\omega}[A_1 + A_2] = j\frac{2.5\times 10^3}{4.33\times 10^3}[2]$$

- Thus

$$A_1 + A_2 = 2$$

$$A_1 - A_2 = j1.155$$

RLC Example Cont'd

- Adding

$$2A_1 = 2 + j1.155$$

$$A_1 = 1 + j0.577$$

- Subtracting

$$2A_2 = 2 - j1.155$$

$$A_2 = 1 - j0.577$$

$$\therefore i(t) = e^{-\alpha t}\left\{\underbrace{\left[e^{j\omega t} + e^{-j\omega t}\right]}_{2\cos\omega t} + 0.577\,\underbrace{j\left[e^{j\omega t} - e^{-j\omega t}\right]}_{2\sin\omega t}\right\}$$

$$i(t) = e^{-2500\,t}\left[2\cos\omega t + 1.155\sin\omega t\right]$$

- Alternate solution - assume

$$i(t) = Ae^{-\alpha t}\cos(\omega t + \theta)$$

1

- At t = 0

$$A\cos\theta = 2$$

$$A = \frac{2}{\cos\theta}$$

RLC Example Cont'd

$$\frac{di}{dt} = -\alpha e^{0} \cos\theta - \omega e^{0} \sin\theta$$

$$= -\alpha \cos\theta - \omega \sin\theta = 0$$

$$\cos\theta = -\frac{\omega}{\alpha} \sin\theta$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = -\frac{\alpha}{\omega}$$

$$\theta = \tan^{-1}(-0.577) = -30°$$

$$A = \frac{2}{\cos(-30°)} = 2.31$$

$$\therefore i(t) = 2.31 e^{-2500\,t} \cos(\omega t - 30°)$$

(c) This was underdamped, for critical damping

$$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$$

thus

$$R = \left(\frac{(2L)^2}{LC}\right)^{\frac{1}{2}} = \left(\frac{4L}{C}\right)^{\frac{1}{2}} = \left(\frac{8 \times 10^{-3}}{10^{-5}}\right)^{\frac{1}{2}}$$

$$R = 40\Omega$$

- For overdamped needs R_1 >40Ω - say 60Ω
- Thus

$$\alpha = \frac{R}{2L} = 7.5 \times 10^{3}\ sec^{-1}$$

- Thus the discriminate

$$\left[\left(\frac{R}{2L} \right)^{2} - \frac{1}{LC} \right] = \left[\left(7.5 \times 10^{3} \right)^{2} - 2.5 \times 10^{7} \right] = 3.125 \times 10^{7}$$

- Thus the solution is

$$s_1 = \alpha + \sqrt{3.125x10^{7}} = 13090\ sec^{-1}$$

$$s_2 = \alpha - \sqrt{3.125x10^{7}} = 1909\ sec^{-1}$$

$$i(t) = \left[A_1 e^{-13090\ t} + A_2 e^{-1909\ t} \right]$$