ENSC 895: SPECIAL TOPICS:
THEORY, ANALYSIS, AND SIMULATION OF NONLINEAR CIRCUITS
FINAL PROJECT PRESENTATIONS

Spring 2004

Study on the stability of nonlinear circuits

Qing Wu
www.sfu.ca/~qingw
Email: qingw@sfu.ca
Outline

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1. Introduction

- Stability is an important characteristic of a dynamic system.
- Checking the stability behavior of a circuit is a crucial stage in circuit design process.
- Stability analysis for circuit basically contains stability analysis of equilibrium point and that of operating point.
2. Theory on stability of circuit

- **Equilibrium point**

  Dynamic system is described by a set of differential equations:

  \[ \dot{x} = f(x) \quad (1) \]

  where \( x \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \). The set of equilibrium points is defined to be

  \[ \{ x : f(x) = 0 \} \quad (2) \]

- It is within the context of a dynamic system.
• If all natural frequencies of a dynamic system are located in the open left half-plane, then the equilibrium point is *stable*.

• If at least one natural frequency is in the open right half-plane then the equilibrium point is *unstable*.
DC operating point

- Voltages and currents across all branches of the static elements in the circuit (e.g. transistors and resistors).

- It is independent of any dynamic elements (e.g. capacitors and inductors).
• A circuit’s operating point is said to be potentially stable if, by inserting some set of positive-valued shunt capacitors and series inductors into the circuit, the corresponding equilibrium point of the resulting dynamic circuit is stable, and if the equilibrium point of any dynamic circuit created by adding this dynamic circuit with an arbitrary set of shunt capacitors and series inductors, whose values are sufficiently small, is also stable.
• Operating point that is not potentially stable is said to be *unstable*. 
In general, any circuit consisting of positive-valued resistors, diodes, transistors (BJT, JFET, MOS, GaAs, etc.) and capacitors and inductors with strictly monotone-increasing charge-voltage or flux-current characteristics, respectively, linearized at a given operating point, can be modeled by the circuit shown in Fig.1.
Fig. 1 general linearized circuit
• the resistive $n$-port $N$ in Figure 1 can be characterized by

$$\begin{bmatrix}
Q_A & Q_B \\
Q_C & Q_D
\end{bmatrix}\begin{pmatrix}
y_d \\
y_o
\end{pmatrix} + \begin{pmatrix}
x_d \\
x_o
\end{pmatrix} = 0$$

(3)

The $n \times n$ matrix $Q$, shown in (3) has a non-negative determinant, since $N$ contains only passive elements.
The port constraints of the circuit in Fig. 1 are given by

$$\begin{pmatrix} y_d \\ y_o \end{pmatrix} = K \frac{d}{dt} \begin{pmatrix} x_d \\ x_o \end{pmatrix} + \begin{bmatrix} A_A & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_d \\ x_o \end{pmatrix}$$ \hspace{1cm} (4)$$

$K$ is a diagonal $n \times n$ matrix whose diagonal elements represent the positive linearized capacitor and inductor values.

$A$ is the $n \times n$ matrix whose elements specify the appropriate dependent source gains. Its upper left-hand $k \times k$ submatrix is nonzero.
Combining (3) and (4), the dynamic equation of the circuit is:

\[ QK \frac{d}{dt} \begin{pmatrix} x_d \\ x_o \end{pmatrix} + (QA + I_n) \begin{pmatrix} x_d \\ x_o \end{pmatrix} = 0 \]  

(5)

The natural frequencies of the circuit are the values of \( s \) that satisfies

\[ \det[sQK + (QA + I_n)] = 0 \]  

(6)

\( \Gamma \) is defined

\[ \Gamma \equiv \det(Q_A A_A + I_k) \]  

(7)
• Theorem 1 ([1], [3]): Given an operating point of a circuit which can be linearized as that in Fig.1 with its dynamic equations written as in (5), if \( \Gamma < 0 \) then the operating point is unstable.

• The outside capacitors and inductors are not necessarily modeled.
Application to transistor circuit

Fig. 2. Nonlinear and linear transistor model

- Linearized at some operating point, with diode conductance $d_1 = \frac{df_1(v_1)}{dv_1}$, and $d_2 = \frac{df_2(v_2)}{dv_2}$. 
Fig. 3. Linear $k$-port circuit
• Assume the linear circuit contains, for \( k \) even, \( k/2 \) transistors. A set of equations for the linear \( k \)-port \( N' \) in the form

\[
Qi' + v = 0
\]  \hspace{1cm} (8)

• The relation between the vectors \( i' \) and \( i \) is

\[
i' = i - Dv
\]  \hspace{1cm} (9)

where \( D = \text{diag}[d_1, \ldots d_k] \).

• The equation describing \( N \) is

\[
Qi + Pv = 0
\]  \hspace{1cm} (10)
where $P = (I_k - QD)$

- The port-variable constraints imposed by the linearized transistors are given by
  \[ i = TDv \quad (11) \]
  where $T$ is defined to be a block diagonal matrix consisting of a set of $2 \times 2$ submatrix of the form
  \[
  \begin{bmatrix}
  1 & -\alpha_m \\
  -\alpha_{m+1} & 1 \\
  \end{bmatrix} \quad 1 \leq m \leq k - 1; m \text{ odd}
  \]
Combining (10) and (11), we have
\[(QTD + P)v = 0\]  \hspace{1cm} (12)

Equivalent expression
\[Q_A = Q\]  \hspace{1cm} (13)
\[A_A = (T-I_k)D\]

So,
\[Q_A A_A + I_k = Q(T-I_k)D + I_k\]
\[= QTD + (I-QD)\]
\[= QTD + P\]  \hspace{1cm} (14)
• **Theorem 2 ([1], [3]):**
  If a transistor circuit has no feedback structure, then its (unique) operating point is potentially stable.

• **Theorem 3 ([1], [3]):**
  If a two-transistor circuit has only one operating point, then it is potentially stable.
• Theorem 4 ([1], [3]):

If a two-transistor circuit has three operating points, then the middle operating point is unstable and the outside operating points are potentially stable.
3. Simulations

- Examples
  1. flip-flop circuit
  2. one circuit in assignment

- Tools & Methods
  PSpice: find operating points
  MATLAB: identify stability of operating point
Example 1

Fig. 4. flip-flop circuit
Fig. 5. flip-flop circuit with connection of two bases
1. Obtain all operating points

- Connect a dc voltage source $V_s$ between the two bases, which destroys original feedback structure. (see Fig. 5)
- Driving point characteristic is shown in Fig. 6.
- The operating points of the original circuit are given by the intersections of the curve with the $V_s$ axis; i.e. the points where $I_s = 0$.
- This two-transistor circuit possesses no more than three operating points.
Fig. 6. Driving point $v-i$ characteristic
Fig. 6. voltages of each node
• PSpice uses Newton-Raphson algorithm to compute the operating point. Without any specification, PSpice sets initial values of all node voltage be zero. The first operating point OP1 is listed as follows and plotted in Fig. 7.

\[ V_1 = 5.0000\text{V}, \ V_2 = 0.9631\text{V}, \ V_3 = 0.9631\text{V} \]

\[ V_4 = 0.7029\text{V}, \ V_5 = 0.7029\text{V}. \]
Fig. 7. first operating point of flip-flop circuit
Set initial values of each node voltage, based on the Fig. 6, for example:

\[ V_1(0) = 0\text{V}, \ V_2(0) = 4.7\text{V}, \ V_3(0) = 0.1\text{V} \]
\[ V_4(0) = 0\text{V}, \ V_5(0) = 0.75\text{V} \]

we get the second operating point OP2:

\[ V_1 = 5.0000\text{V}, \ V_2 = 4.6105\text{V}, \ V_3 = 0.0861\text{V} \]
\[ V_4 = 0.0861\text{V}, \ V_5 = 0.7156\text{V} \]

It is shown in Fig. 8.
Fig. 8. second operating point of flip-flop circuit
Similarly, setting $V_1(0) = 0\text{V}$, $V_2(0) = 0.1\text{V}$, $V_3(0) = 4.7\text{V}$, $V_4(0) = 0.75\text{V}$, $V_5(0) = 0\text{V}$. The third operating point OP3 is

$V_1 = 5.0000\text{V}$

$V_2 = 0.0861\text{V}$

$V_3 = 4.6105\text{V}$

$V_4 = 0.7156\text{V}$

$V_5 = 0.0861\text{V}$

It is shown in Fig. 9.
Fig. 9. third operating point of flip-flop circuit
2. Identify stability of each operating point

2.1 Write circuit equation in the form

\[ P_v + Q_i = b \]  \hspace{1cm} (15)

or

\[ G_v + i = b' \]  \hspace{1cm} (16)

2.2 Set parameters

From PSpice

Ideal maximum forward beta \( \beta_f = 416.4 \).

Ideal maximum reserve beta \( \beta_r = 0.7371 \).
So, $\alpha_r$ and $\alpha_f$ in matrix T are

$$\alpha_r = \beta_r / (1 + \beta_r)$$

$$\alpha_f = \beta_f / (1 + \beta_f)$$

$$f(v_i) = m_i (\exp(nv_i) - 1)$$

for emitter, $m$ is reserve saturation current of the base-emitter junction, $m = I_s / \alpha_f$.

for collector, $m$ is reserve saturation current of the base-collector junction, $m = I_s / \alpha_r$.

$I_s$ is the $p$-$n$ saturation current. Here, for two transistors $I_{s1} = I_{s2} = 6.734e-15$. 
• $n$ is a constant at a certain temperature. $n = \frac{q}{kT}$, where $q$ is electron charge, $k$ is Boltzmann’s constant and $T$ is temperature (in Kelvin).

• For PNP transistor, $m, n > 0$, and for NPN transistor $m, n < 0$.

• D is a diagonal matrix with elements $d_1 \sim d_4$.
  \[ d_i = m_i n \exp(nv_i) \quad (i = 1, 2, 3, 4) \]
2.3 Compute $\det(Q_{TD}+P)$ and observe the sign of $\Gamma$.

OP1: $\Gamma_1 = \det(Q_{TD} + P) = -2.8077\text{e}-10 < 0$

OP2: $\Gamma_2 = \det(Q_{TD} + P) = 4.0475\text{e}-10 > 0$

OP3: $\Gamma_3 = \det(Q_{TD} + P) = 4.0475\text{e}-10 > 0$

So, the middle operating point OP1 is unstable and two outside operating points OP2 and OP3 are both potentially stable.
Example 2

Fig. 10 circuit in assignment
Fig. 11 circuit in PSpice
Fig. 12 port current and node voltage
1. Obtain all operating points

From the $\nu$-$i$ characteristic, when the circuit is driven by a current source, it may exist three operating points. (i.e. $I = 200mA$).

Set driving source to be 200mA current source, and use PSpice to simulate it. The results are shown in Fig. 13 to Fig. 15.
Fig. 13. first operating point of circuit
Fig. 14. second operating point of circuit
Fig. 15 third operating point of circuit
2. Identify stability of each operating point

- Write circuit equations.
- Set parameters in transistors in the same way with that in example 1.

- Compute $\Gamma$ and the results are:
  - OP1: $\Gamma_1 = \det(QTD+P) = -0.7546 < 0$
  - OP2: $\Gamma_2 = \det(QTD+P) = 1.2171 \times 10^3 > 0$
  - OP3: $\Gamma_3 = \det(QTD+P) = 0.0734 > 0$
• The middle operating point OP1 is unstable and two outside operating points OP2 and OP3 are both potentially stable.
4. Future work

- Study more complex circuit such as Schmitt trigger circuit and those in assignment.
- Apply homotopy method to obtain all operating points simultaneously and set the initial values of node voltage based on the result.
- Apply degree theory to the identification of unstable operating point.
References


