Spectral Analysis and Dynamical Behavior of Complex Networks

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Abstract—In this paper, we employ spectral graph theory as a tool for analyzing the Internet topology. We show its importance in understanding dynamical behavior of complex networks. We also provide an overview of various approaches dealing with synchronization in complex networks.

1. Introduction

A variety of complex networks has been identified in real life. Many have universal characteristics such as small-world [1] and scale-free [2] topologies.

Analysis of complex networks often relies on discovering spectral properties of graphs capturing their topology. Such analysis is based on constructing matrices describing the network connectivity. Both the well-known adjacency matrix (also called Kirchhoff matrix) and variants of the Laplacian matrix (including normalized Laplacian and signless Laplacian matrices) of graphs capturing network structure have been employed in such analysis.

We describe analysis of large datasets collected from the Internet over several years. Spectral analysis of graphs constructed from these datasets confirms the existence of power-laws and was used to identify historical trends in the development of the network. Spectral analysis of the associated graphs also reveals historical trends in the clustering of network nodes and their connectivity. These connectivity and clustering properties of the network may be further analyzed by examining element values of the corresponding eigenvectors.

Dynamics in complex networks has recently attracted considerable research interest stimulated by the study of synchronization in systems with multiple oscillators. In 1970s, analysis of network dynamics was related to electrical networks. In this paper, we provide an overview of analysis of network dynamics with particular attention to synchronization.

2. The Internet Topology

Analyzing the Internet topology using randomly generated graphs, where routers are represented by vertices and transmission lines by edges, has been widely replaced by mining data that capture information about Internet Autonomous Systems and by exploring properties of associated graphs on the AS-level. The Route Views data [3] and RIPE [4] datasets collected from Border Gateway Protocols (BGP) routing tables have been extensively used by the research community [5]–[7]. The discovery of power-laws and spectral properties of the Internet topology indicates a complex underlying network infrastructure.

Analysis of the collected datasets indicates that the Internet topology is characterized by the presence of various power-laws observed when considering a node degree vs. node rank, a node degree frequency vs. degree, and a number of nodes within a number of hops vs. number of hops [5], [6]. Some of these early conclusions were subsequently revised by considering a more complete AS-level representation of the Internet topology [7], [8]. These extended maps have heavy tailed or highly variable degree distributions and only the distribution tales have the power-law property. It has been observed that the power-law exponents associated with Internet topology have not substantially changed over the years in spite of the Internet exponential growth [9]–[11]. Power-laws also appear in the eigenvalues of the adjacency matrix and the normalized Laplacian matrix vs. the order of the eigenvalues. They also show invariance regardless of the exponential growth of the Internet.

While various power-law exponents associated with the Internet topology have remained similar over the years, indicating that the power-laws do not capture every property of a graph and are only one measure used to characterize the Internet, spectral analysis of both the adjacency matrix and the normalized Laplacian matrix of the associated graphs reveals new historical trends in the clustering of AS nodes and their connectivity. The eigenvectors corresponding to the largest eigenvalues of the normalized Laplacian matrix have been used to identify clusters of AS nodes with certain characteristics [9]. Spectral analysis was employed to analyze the Route Views and RIPE datasets in order to find distinct clustering features of the Internet AS nodes [12]. For example, the connectivity graphs of these datasets indicate visible changes in the clustering of AS nodes and the AS connectivity over the period of five years [10]. Clusters of AS nodes can be also identified based on the eigenvectors corresponding to the second smallest and the largest eigenvalue of the adjacency matrix and the normalized Laplacian matrix [11]. The connectivity and clustering properties of the Internet topology can be further analyzed by examining element values of the corresponding eigenvectors.

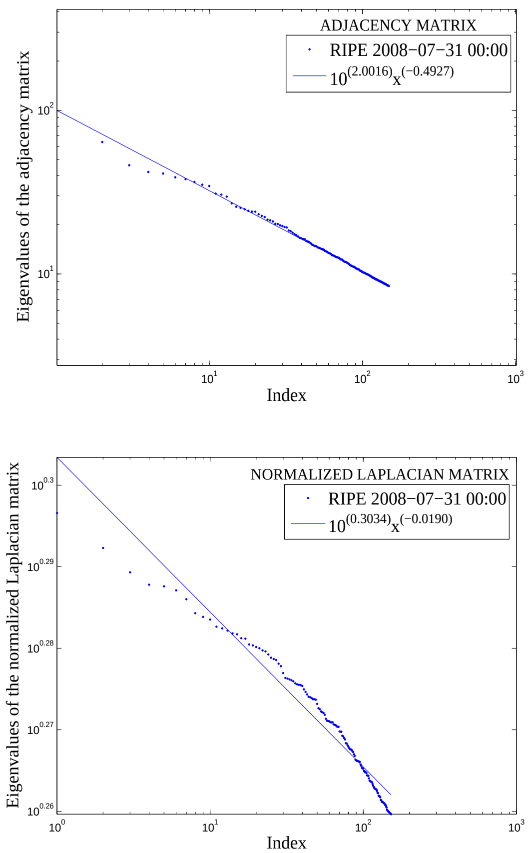

Figure 1: RIPE 2008 dataset: The eigenvalue power-law exponents for the adjacency matrix (top) is –0.4927 with the correlation coefficients –0.9970. The eigenvalue power-law exponents for the normalized Laplacian matrix (bottom) is –0.0190 with the correlation coefficients –0.9758.

An example of the dependencies between the graph eigenvalues and the eigenvalue index are shown in Fig. 1 [11]. Plotted on a log-log scale are eigenvalues in decreasing order. Only the 150 largest eigenvalues are plotted.

3. Dynamics in Complex Networks

Earlier analysis of network dynamics addressed regular networks (nodes have equal degree) [13], [14], [15], where synchronism in the lattice, ladder, and ring networks were discussed and the conditions of complete synchronism were derived. In these papers, each node contained a Van der Pol oscillator and was connected to other nodes by resistors or inductors. In 1990's, the complex phenomena in networks with chaotic circuits was intensively analyzed [16]–[19]. [18], [19]. In electrical systems, star-connected oscillators [20] and the ring coupling of chaotic circuits were analyzed [21]. Coupled oscillators networks were also analyzed in the context of cellular neural networks [22].

3.1. Synchronization in Small-World Networks

Small-world networks have two main properties: small average distance $\overline{D}$ and high clustering. Some use only the first property as the definition of small-world networks. The answer to the question whether or not synchronization is easily achieved on a network with small-world property is somewhat surprising: The small-world property does not generally guarantee synchronization in the network [23].

3.2. Synchronization in Scale-Free Networks

Scale-free networks are characterized by the power-law connectivity distribution of certain network variables such as, for example, $P(k) \propto k^{-\gamma}$, where $P(k)$ the probability distribution function and k is the node degree of the network. The smaller the parameter γ, the more the network becomes heterogeneous in its connectivity distribution and, accordingly, the average network distance decreases. However, when the average network distance becomes smaller, synchronization is more difficult to achieve. This result was explained by considering the load (information) on center nodes (hubs), of a network [24].

3.3. Synchronization in Complex Networks

In recent years, dynamical behavior of complex networks has been of particular interest. Each node in a complex network contains an oscillator or a dynamical system that generates periodic or chaotic oscillations. The network topology is represented by a Laplacian matrix $L(G)$, which is symmetric and has a single zero eigenvalue for a connected network.

The number of edges incident to a node in an undirected graph is called the degree of the node. Two nodes are called adjacent if they are connected by a link. The network can be represented by the adjacency matrix A(G):

$$A(i, j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

A diagonal matrix D(G) associated with A(G), with row-sums of A(G) as the diagonal elements, indicates the connectivity degree of each node. The Laplacian matrix is defined as L(G) = D(G) – A(G). The eigenvalues of L(G) are closely related to certain graph invariants. The spectrum of $L(G)$ is the collection of all eigenvalues and contains 0 for every connected graph component.

The general synchronization condition for complex networks with a large number of oscillators was derived in [25]. We provide here a brief overview of the employed analytical method.

3.3.1. Master Stability Equation and Master Stability Function

We consider a network with N nodes and assume that each network node is governed by a self-oscillatory autonomous system with m variables. For example, $m = 2$

in case of the Van der Pol oscillator and $m = 3$ for the Lorenz system. We assume that the oscillators are identical with identical coupling to other oscillators.

- We first formulate the state equation of the network by introducing Laplacian matrix. For very large dimensional spaces, the direct product offers a convenient expression. In order to show the overall coupling, a constant σ is introduced.

  Let $\mathbf{x}^i$ be the m-dimensional vector of state variables of the i-th node. Let $\mathbf{F}(\mathbf{x}^i)$ be the isolated (uncoupled) dynamics for each node and $\mathbf{H}$: $R^m \rightarrow R^m$ be a coupling function. The dynamics of node i can be expressed as:

$$\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) + \sigma \sum_{j=1, j\neq i}^{N} G_{ij}\mathbf{H}(\mathbf{x}^j), \qquad (1)$$

  where σ is a coupling strength.

  We define matrices $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N)$, $\mathbf{F}(\mathbf{x}) = [\mathbf{F}(\mathbf{x}^1), \mathbf{F}(\mathbf{x}^2), \cdots, \mathbf{F}(\mathbf{x}^N)]$, and $\mathbf{H}(\mathbf{x}) = [\mathbf{H}(\mathbf{x}^1), \mathbf{H}(\mathbf{x}^2), \cdots, \mathbf{H}(\mathbf{x}^N)]$. Let $\mathbf{G}$ be the NxN matrix of coupling coefficients G_{ij}. Note that $\mathbf{G} = -L(G)$. The dynamics of the network is described as:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \sigma\mathbf{G} \otimes \mathbf{H}(\mathbf{x}), \qquad (2)$$

  where $\otimes$ is the direct product.

- We then seek to find the periodic solutions of the state equation (1).

- We derive the variational equation from the periodic steady-state in order to investigate the stability of synchronized steady-state or periodic solution

$$\dot{\xi} = [\mathbf{1}_N \otimes D\mathbf{F} + \sigma\mathbf{G} \otimes D\mathbf{H}]\xi, \qquad (3)$$

  where ξ^i are variations on node i and $\xi = (\xi^1, \xi^2, \cdots, \xi^N)^{tr}$.

  The variational equation becomes the linear differential equation with periodic coefficients combined with the Laplacian matrix. By using an appropriate linear transformation, the variational equation can be divided in separate blocks, each block corresponding to an eigenvalue $\gamma_k (k = 0, \cdots, N-1)$, where N is the number of nodes:

$$\dot{\xi}_k = [D\mathbf{F} + \sigma\gamma_k D\mathbf{H}]\xi_k. \qquad (4)$$

  Each separate block equation is called the *master stability equation*.

- From the variational equation, we compute the maximum Lyapunov exponents Λ_{max} called the *master stability function*. If Λ_{max} is negative, the corresponding periodic steady-state is stable and the variations die out. The stability investigation is the extension of the 2nd order nonlinear differential equation such as the Duffing's equation [26].

Factor $\alpha \equiv \sigma\gamma_k$, defined as the product of γ_k and the overall strength of coupling parameter σ, is a measure used to express the coupling strength. The stability plots of Λ_{max} vs. α (generic coupling factor for nonlinear function and output function at each node) are used to define stability regions. The oscillatory systems such as periodic oscillators have a master stability function that has $\Lambda_{max} < 0$ over the interval $(\alpha_{min}, \alpha_{max})$ in these stability plots. The generic requirement for the synchronous state to be stable is given by $\sigma\lambda_k \in (\alpha_{min}, \alpha_{max})$ for each k. This requirement can be equivalently written as $\lambda_{max}/\lambda_1 < \alpha_{max}/\alpha_{min}$, where λ_1 and λ_{max} are the second smallest and the largest eigenvalues, respectively [23]. The left-hand side of the inequality is determined solely by the Laplacian matrix while the right-hand side is defined by the master stability function. Hence, we can analyze the stability of synchronization and network dynamics by only observing the network topology.

3.3.2. The Internet Dynamics

In case of Internet graphs, the Laplacian matrix has distinct real eigenvalues $\lambda_k (k = 0, \cdots, N-1)$, where N is the number of network nodes [27]. The behavior of the nodes is governed by network transport protocols and queuing algorithms. Hence, the network dynamics can be described by using the fluid-model of the Transport Control Protocol (TCP) combined with the Random Early Detection (RED) queuing algorithm [28]:

$$\begin{aligned}
\frac{dw(t)}{dt} &= \frac{1}{r(t)} - \frac{w(t)}{2}\frac{w(t-r(t))}{r(t-r(t))}p(t-r(t)) \\
\frac{dq(t)}{dt} &= N\frac{w(t)}{r(t)} - C \\
\frac{dx(t)}{dt} &= C\ln(1-\alpha)(x(t)-q(t))
\end{aligned} \qquad (5)$$

$$p_b(t) = \begin{cases} 0 & 0 \leq x(t) < x_{\min} \\ \dfrac{x(t)-x_{\min}}{x_{\max}-x_{\min}}p_{\max} & x_{\min} \leq x(t) \leq x_{\max} \\ p_{\max} & \\ +\dfrac{1-p_{\max}}{x_{\max}}(x(t)-x_{\max}) & x_{\max} < x(t) \leq 2x_{\max} \\ 1 & 2x_{\max} \leq x(t) \end{cases} \qquad (6)$$

$$p(t) = \kappa p_b(t) \text{ and } r(t) = \frac{q(t)}{C} + R_0, \qquad (7)$$

where
$w(t)$ = averaged instantaneous window size (in packets) of the TCP sources
$r(t)$ = round trip time
$q(t)$ = averaged instantaneous queue length (in packets)
$x(t)$ = filtered queue length after removal of short bursts
$p(t)$ = marking probability
α = filter resolution ($0 < \alpha < 1$)
κ = a proportionality constant dependent on the implementation of the RED algorithm
$x_{\max}$ = maximum threshold of $x(t)$

$x_{\min}$ = minimum threshold of $x(t)$
$p_{\max}$ = maximum threshold of $p(t)$
R_0 = propagation delay
C = bottleneck bandwidth in packets/second
B = maximum physical queue length.

4. Conclusions and Future Work

In this paper, we consider numerous new aspects of the dynamics of complex networks and we do not necessarily restrict our attention to classical small-world and scale-free networks. Included in this analysis are also many electrical networks such as regular networks. One of the problems that we plan to address is universal quantification using differential equations combined with graph theory. In various applications of complex networks, it is essential to deal with dynamics of complex networks with the weights imposed on network nodes and edges. Furthermore, we plan to develop effective methods for obtaining synchronous solutions of nonlinear equations with higher dimensions.

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References

[1] D. J. Watts and S. H. Strogatz, "Collective dynamics of small world networks," *Nature*, vol. 393, pp. 440-442, June 1998.

[2] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, pp. 509–512, October 1999.

[3] (2010, February 10). BGP capture datasets [Online]. Available: http://archive.routeviews.org.

[4] (2010, February 10). Réseaux IP Européens [Online]. Available: http://www.ripe.net/ris.

[5] M. Faloutsos, P. Faloutsos, and C. Faloutsos, "On power-law relationships of the Internet topology," in *Proc. ACM SIGCOMM, Computer Communication Review*, vol. 29, no. 4, pp. 251–262, Sept. 1999.

[6] G. Siganos, M. Faloutsos, P. Faloutsos, and C. Faloutsos, "Power-laws and the AS-level Internet topology," *IEEE/ACM Trans. Networking*, vol. 11, no. 4, pp. 514–524, Aug. 2003.

[7] Q. Chen, H. Chang, R. Govindan, S. Jamin, S. Shenker, and W. Willinger, "The origin of power laws in Internet topologies revisited," in *Proc. INFOCOM*, New York, NY, USA, Apr. 2002, pp. 608–617.

[8] H. Chang, R. Govindan, S. Jamin, S. Shenker, and W. Willinger, "Towards capturing representative AS-level Internet topologies," in *Proc. ACM SIGMETRICS Performance Evaluation Review*, New York, NY, June 2002, vol. 30, no. 1, pp. 280–281.

[9] C. Gkantsidis, M. Mihail, and E. Zegura, "Spectral analysis of Internet topologies," in *Proc. INFOCOM*, San Francisco, CA, Mar. 2003, vol. 1, pp. 364–374.

[10] M. Najiminaini, L. Subedi, and Lj. Trajković, "Analysis of Internet topologies: a historical view," *Proc. IEEE Int. Symp. Circuits and Systems*, Taipei, Taiwan, May 2009, pp. 1697–1700.

[11] L. Subedi and Lj. Trajković, "Spectral analysis of Internet topology graphs," *IEEE Int. Symp. Circuits and Systems*, Paris, France, May 2010, pp. 1803–1806.

[12] J. Chen and Lj. Trajković, "Analysis of Internet topology data," in *Proc. IEEE Int. Symp. Circuits Syst.*, Vancouver, BC, Canada, May 2004, vol. IV, pp. 629–632.

[13] T. Endo and S. Mori, "Mode analysis of a multi-mode ladder oscillator," *IEEE Trans. on Circuits and Systems*, vol. CAS-23, no. 2, pp. 100–113, 1976.

[14] T. Endo and S. Mori, "Mode analysis of a two-dimensional low-pass multi-mode oscillator," *IEEE Trans. on Circuits and Systems*, vol. CAS-23, no. 9, pp. 517–530, 1976.

[15] T. Endo and S. Mori, "Mode analysis of a ring of a large number of mutually coupled van der Pol oscillators," *IEEE Trans. on Circuits and Systems*, vol. CAS-25, no. 1, pp. 7–18, 1978.

[16] Y. Nishio and A. Ushida, "On a ring of chaotic circuits coupled by inductors," *IEICE Trans. on Fundamentals*, vol. E78-A, no. 5, pp. 608–617, 1995.

[17] Y. Nishio and A. Ushida, "Spacio-temporal chaos in simple coupled chaotic circuits," *IEEE Trans. on Circuits Syst. I*, vol. CAS-42, no. 10, pp. 678–686, 1995.

[18] Y. Nishio and A. Ushida, "Quasi-synchronization phenomena in chaotic circuits coupled by one resistor," *IEEE Trans. on Circuits Syst. I*, vol. CAS-43, no. 6, pp. 491–496, 1996.

[19] Y. Nishio and A. Ushida, "Chaotic wandering and its analysis in simple coupled chaotic circuits," *IEICE Trans. on Fundamentals*, vol. E85-A, no. 1, pp. 248–255, 2002.

[20] S. Moro, Y. Nishio, and S. Mori, "Synchronization phenomena in oscillators coupled by one resistor," *IEICE Trans. on Fundamentals*, vol. E78-A, no. 2, pp. 244–253, 1995.

[21] Y. Uwate and Y. Nishio, "Complex phase synchronization in an array of oscillators coupled by time-varying resistor," *Int. Joint Conference on Neural Networks*, 2006, pp. 8345–835.

[22] S. Moro, Y. Nishio, and S. Mori, "On coupled oscillators networks for cellular neural networks," *IEICE Trans. Fundamentals*, vol. E80-A, no. 1, pp. 214-222, 1997.

[23] M. Barahona and L. M. Pecora, "Synchronization in small-world systems," *Physical Review Letters*, vol. 89, no. 5, pp. 054101-1–4, 2002.

[24] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, "Heterogeneity in oscillator networks: are smaller worlds easier to synchronize?," *Physical Review Letters*, vol. 91, pp. 014101, 2003.

[25] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Physical Review Letters*, vol. 80, no. 10, pp. 2109–2112, 1998.

[26] C. Hayashi, *Nonlinear Oscillations in Physical Systems*. New York: McGraw-Hill, 1964, pp. 82–97.

[27] F. R. K. Chung, *Spectral Graph Theory*. Providence, Rhode Island: Conference Board of the Mathematical Sciences, 1997, pp. 2–6.

[28] V. Misra, W. B. Gong, and D. Towsley, "Fluid-based analysis of a network of AQM routers supporting TCP flows with an application to RED," in *Proc. ACM/SIGCOMM*, Stockholm, Sweden, Aug. 2000, pp. 151–160.