ENSC383: Feedback Control Systems
School of Engineering Science, Simon Fraser University
Final Exam, Dec 5, 2007 2007, Location: EDB 7618
Time: ***12 to 2:30pm***

- This exam has a weight of 50% of the overall mark. Questions are market out of 50.
- The exam is closed-book. A two page crib sheet without any solved problems is permitted which has to be handed in with the exam paper.
- Please write legibly and show your work. You may lose marks if your work is not legible or not clear. Where needed make appropriate assumptions and explain.

Name:____________________________ Student I.D.: _______________

(6 marks) Q1. Comment on the stability of each system with the transfer function given below. Give ~~a~~ valid reasons for your answers:

(a) $H(s) = \frac{s+1}{s^3+s^2-9s-9}$

Unstable. $s^3 + s^2 - 9s - 9$

indicates right half plane roots

Can also be shown by Routh's test

(b) $G(s) = \frac{s-2}{(s^2+2s+2)(s+1)}$

Roots $\begin{cases} s^2+2s+2=0 \longrightarrow & s = -1 \pm \sqrt{1-2} = -1 \pm j1 \\ s=-1 & \end{cases}$

$\longrightarrow$ Stable!

(c) $T(s) = \frac{s+8}{(s^2+25)^2\,(s+10)}$

$(s^2+25)^2=0 \longrightarrow \quad s = \pm j5$ of multiplicity 2

$s+10=0 \longrightarrow \quad s=-10$

System is not stable!

(7 marks) Q2. Consider a system with transfer function $G(s) = \frac{25}{25s^2+20s+25}$

(a) Find the settling time and percent overshoot when a unit step input is applied to the system.

$$G(s) = \frac{1}{s^2 + 0.8s + 1} \longrightarrow \begin{cases} 2\zeta\omega_n = 0.8 \\ \omega_n = 1 \end{cases} \longrightarrow \boxed{\begin{array}{c} \zeta = 0.4 \\ \hline \omega_n = 1 \end{array}}$$

1% settling time $\longrightarrow t_s \approx \frac{4.6}{\zeta\omega_n} = \frac{4.6}{0.4} = 11.5$ sec.

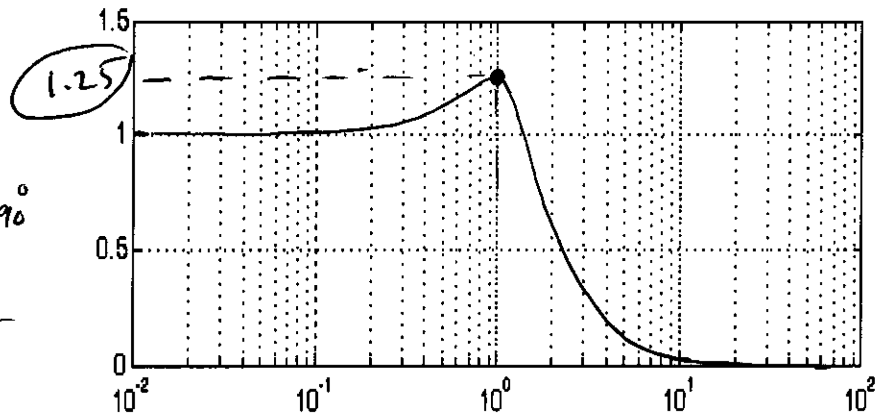
Percent overshoot $\longrightarrow M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = e^{-1.3711} = 0.2538$

OR 25.38% overshoot

(b) Draw the approximate frequency response below showing important points.

$$G(j\omega) = \frac{1}{-\omega^2 + j0.8\omega + 1} = \frac{1}{1-\omega^2+j0.8\omega}$$

ω	$G(j\omega)$
0	1
1	$\frac{1}{j0.8} = 1.25\angle{-90^\circ}$
10	
∞	$0\angle{-180^\circ}$



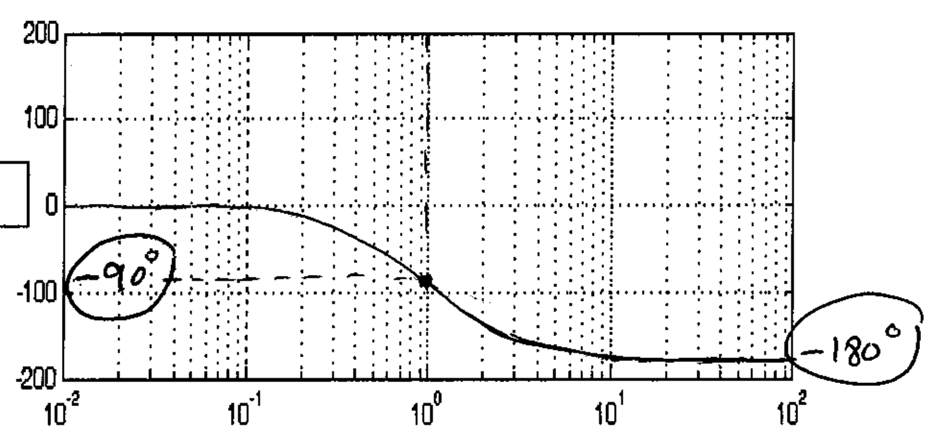
$\omega = 10 \longrightarrow \frac{1}{1-100+j8}$

$= \frac{1}{-99+j8}$

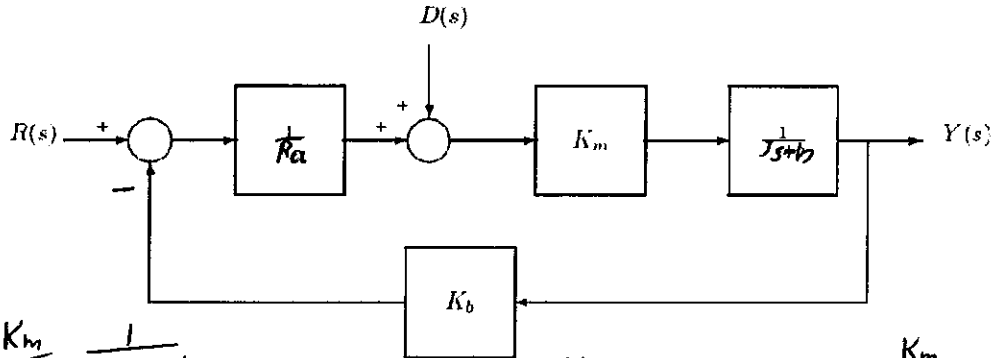
$= \frac{1}{99.32\angle{175.3^\circ}}$

$= 0.0101\angle{-175.3^\circ}$

degrees



(8 marks) Q3. Consider the model of a DC motor shown below where *Y(s)* is the motor speed, *R(s)* is the input voltage, and *D(s)* is the load torque disturbance. Other parameters are motor constants. Show how a Proportional Integral (*PI*) controller can be used to reject step disturbance torques and achieve zero steady state error for step speed commands.

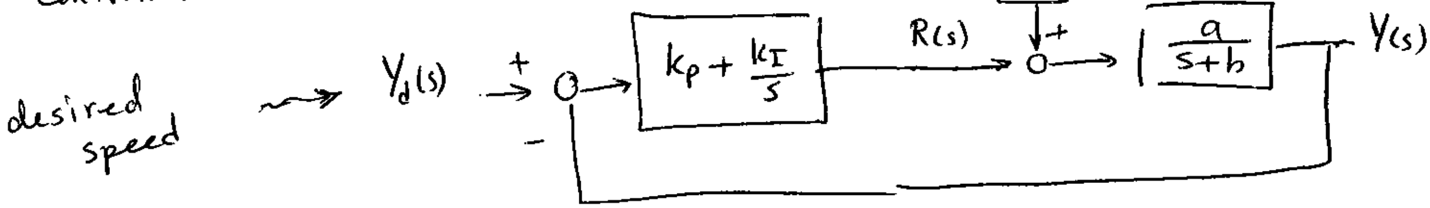


$$\frac{Y(s)}{R(s)} = \frac{\frac{K_m}{R_a}\frac{1}{Js+b}}{1+\frac{K_b K_m}{R_a}\frac{1}{Js+b}} = \frac{K_m}{R_a(Js+b)+K_b K_m} = \frac{\frac{K_m}{R_a J}}{s+\frac{K_b K_m + R_a b}{R_a J}}$$

$$\frac{Y(s)}{D(s)} = \frac{K_m\frac{1}{Js+b}}{1+\frac{K_b K_m}{R_a(Js+b)}} = \frac{K_m R_a}{R_a(Js+b)+K_b K_m} = \frac{\frac{K_m \cancel{R_a}}{\cancel{R_a} J}}{s+\frac{K_b K_m + R_a b}{R_a J}}$$

Thus letting $\frac{Y(s)}{R(s)} = \frac{a}{s+b}$ we have $\frac{Y(s)}{D(s)} = \frac{R_a\, a}{s+b}$

* The closed-loop system with PI controller is illustrated below



Closed loop system poles → $1+\frac{k_p s + k_I}{s}\,\frac{a}{s+b} = 0$

$$s^2 + bs + a k_p s + a k_I = 0 \rightarrow s^2 + (a k_p + b)s + a k_I = 0$$

Thus k_p & k_I can be used to achieve arbitrary pole locations.

Since we have one integrator in the loop, the error terms for step input are $\infty \Longrightarrow$ Zero step error and disturbance rejection

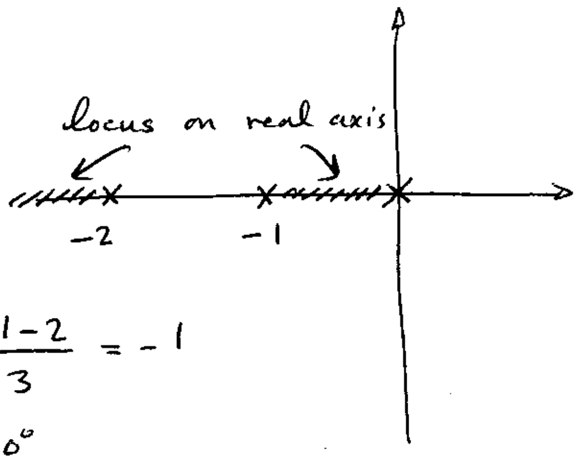
(14 marks) Q4. Consider a system whose characteristic polynomial is given by $D(s) = s^3 + 3s^2 + 2s + k$

Analyze the stability of the system when k varies from 0 to $+\infty$ using the following methods:

(a) Root locus (specify marginal stability points)

$$s^3 + 3s^2 + 2s + k = 0 \longrightarrow 1 + \frac{k}{s(s^2+3s+2)} = 0$$

$$\text{OR} \quad 1 + \frac{k}{s(s+1)(s+2)} = 0$$



Poles : $s = 0,\ s = -1,\ s = -2$

Asymptotes

$$\text{intercept point} = \frac{\Sigma\text{poles} - \Sigma\text{zeros}}{3-0} = \frac{-1-2}{3} = -1$$

$$\text{angles} = \frac{180 + 360l}{3} = 60^\circ, -60^\circ, 180^\circ$$

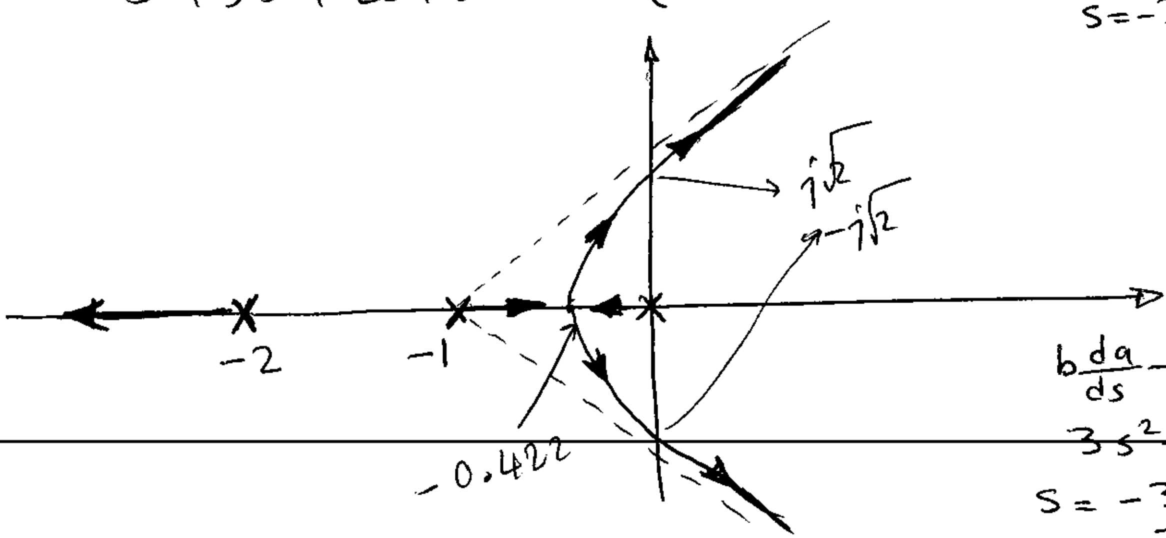
Crossing imaginary axis $\quad s^3 + 3s^2 + 2s + K$

Routh's array $\longrightarrow$

1	2	$\longrightarrow$ $k > 0 \longrightarrow k = 6$
3	k	$K < 6 \longrightarrow k = 0$
$\frac{6-k}{3}$	0	Crossing $j\omega$ axis
k		

$$s^3 + 3s^2 + 2s + 6 = s^2(s+3) + 2(s+3) = (s+3)(s^2+2) = 0$$

$$s = -3,\quad s = \pm j\sqrt{2}$$



acceptable

$$b\frac{da}{ds} - a\frac{db}{ds} = 0$$

$$3s^2 + 6s + 2 = 0$$

$$s = \frac{-3 \pm \sqrt{9-6}}{3} \quad -0.422,\ -1.577$$

... Q4 continued

$$1+\frac{K}{s(s+1)(s+2)} \rightsquigarrow \frac{1}{j\omega(j\omega+1)(j\omega+2)} = G(j\omega)$$

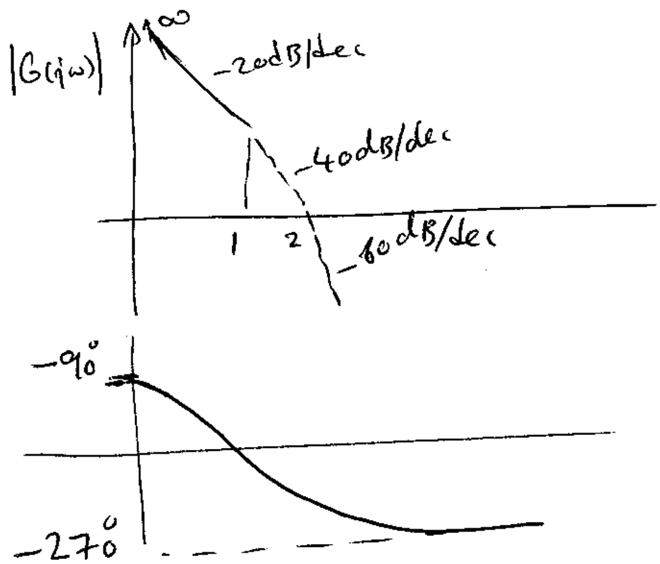
(b) Nyquist analysis

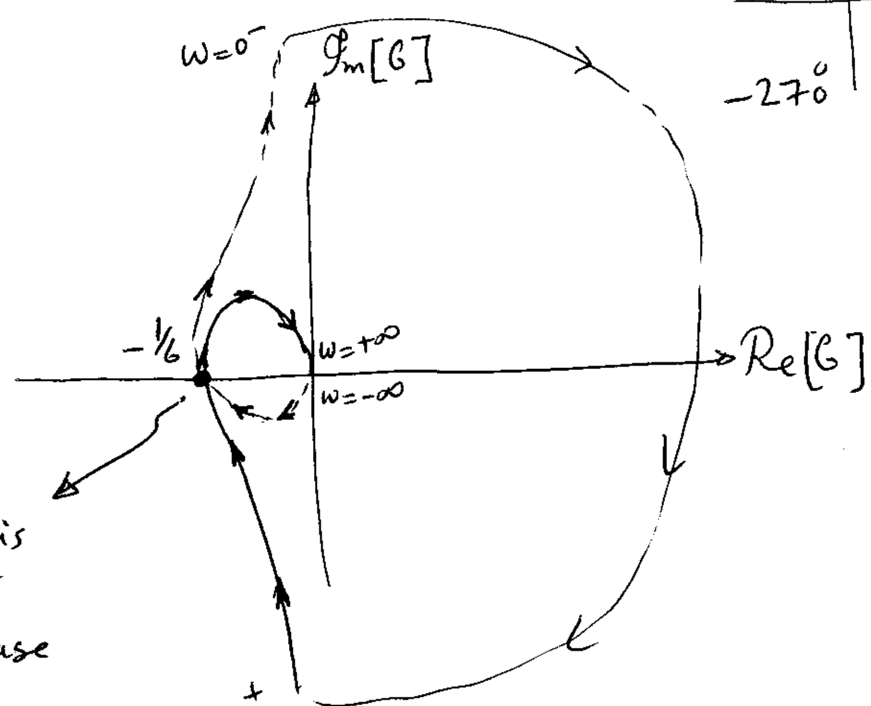
$\omega = 0^+ \rightarrow |G(j\omega)| \rightarrow \infty$

$\angle G(j\omega) = -90^\circ$

$\omega = +\infty \rightarrow |G| \rightarrow 0$

$\angle G \rightarrow -270^\circ$





At this point the phase of G is -180° $\rightarrow$ $G(j\omega) = \dfrac{1}{j\omega(-\omega^2 + j3\omega + 2)}$

$$= \frac{1}{-j\omega^3 - 3\omega^2 + j2\omega}$$

$$= \frac{1}{j(2\omega - \omega^3) - 3\omega^2}$$

Thus $\omega^2 = 2 \rightarrow \omega = \sqrt{2}$

we have:

$$\boxed{G(j\sqrt{2}) = \frac{1}{-3\times 2} = -\frac{1}{6}}$$

$\cancel{Z = N+P}$

If $-\frac{1}{K} < -\frac{1}{6} \Rightarrow Z = N+P = 0$

$0 < K < 6 \Rightarrow$ stable!

If $-\frac{1}{6} < -\frac{1}{K} < 0 \rightarrow N = 2$

$\therefore Z = 2 + 0 = 2$

$\therefore$ Unstable with 2 RHP poles

$K > 6 \Rightarrow$ Unstable

MATLAB Code

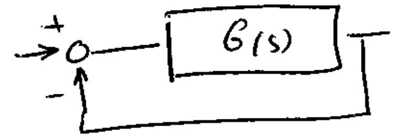
```
num=[1 -4 13]
den=[1 3 2 0]
sys=tf(num,den)
rlocus(sys)
```

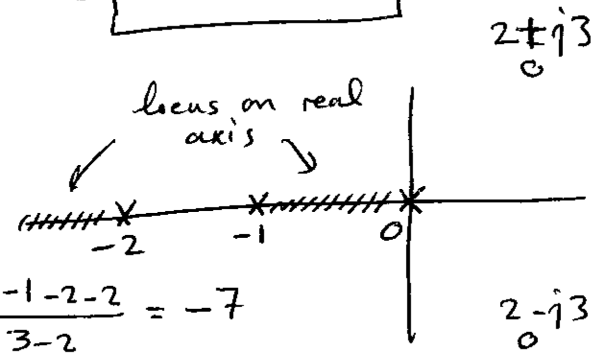
2 marks

2

(7 marks) Q5. Consider a unity negative feedback system with open loop transfer function $G(s) = k(s^2 - 4s + 13)/(s^3 + 3s^2 + 2s)$. Draw the root locus of the system when k>0 is varied from 0 to infinity. Find the asymptotes and give an expression for the angles of arrival of the locus at the zeros. Also write Matlab code for root locus.

$$1 + k\frac{s^2-4s+13}{s(s^2+3s+2)} = 0 \leftarrow \text{ch. eq.}$$





Poles : $s=0,\ s=-2,\ s=-1$

Zeros : $s_1, s_2 = 2 \pm \sqrt{4-13} = 2 \pm j3$

Asymptotes : $\text{intercept point} = \dfrac{\Sigma\text{poles} - \Sigma\text{zeros}}{n-m} = \dfrac{-2-1-2-2}{3-2} = -7$

$$\text{angle} = \frac{180+360l}{n-m} = \pm 180^\circ$$

Imaginary axis crossing : $s^3+3s^2+2s + ks^2-4ks+13k = 0$

$$s^3 + (3+k)s^2 + (2-4k)s + 13k = 0$$

1	$2-4k$
$3+k$	$13k$
$\dfrac{-4k^2-23k+6}{3+k}$	0
$13k$	

$$\rightsquigarrow 4k^2+23k-6=0 \rightarrow k = \frac{-23 \pm \sqrt{23^2+4\times4\times6}}{8} = \frac{1}{4}$$

$a(s) = s^3+3s^2+2s \qquad b = s^2-4s+13$

Breakaway - break-in $\rightarrow b\dfrac{da}{ds} - a\dfrac{db}{ds} = 0 \rightarrow (s^2-4s+13)(3s^2+6s+2) - (s^3+3s^2+2s)(2s-4) = 0$

$$3s^4 + \cancel{6s^3} + 2s^2 - 12s^3 - 24s^2 - 8s + 39s^2 + 78s + 26$$
$$-2s^4 + 4s^3 - \cancel{6s^3} + 12s^2 - 4s^2 + 8s = 0$$

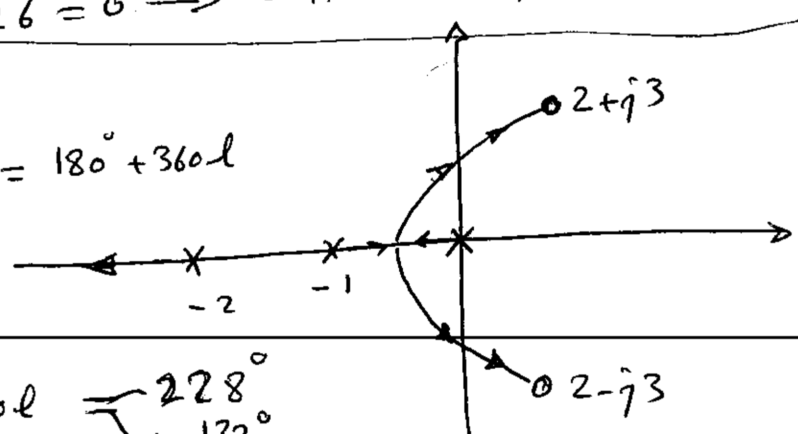
$s^4 - 8s^3 + 21s^2 + 78s + 26 = 0 \rightarrow$ difficult to find roots

Not necessary

Angle of arrival at zeros

$$\phi_{arr} + 90^\circ - \underbrace{\tan^{-1}\frac{3}{2}}_{\text{pole at } s=0} - \underbrace{\tan^{-1}\frac{3}{3}}_{\text{pole at } s=-1} - \underbrace{\tan^{-1}\frac{3}{4}}_{\text{pole at } s=-2} = 180^\circ + 360l$$

$56.3^\circ \qquad 45^\circ \qquad 36.8^\circ$



$$\phi_{arr} = 90 + 56.3^\circ + 45^\circ + 36.8^\circ + 360l = 228^\circ,\ 132^\circ$$

(8 marks) Q6. Consider a unity feedback control system consisting of a plant with transfer function $G(s) = \frac{1}{s^2}$.

(a) Design a lead compensator $D(s) = K\,(s+z)/(s+p)$ to be added in series with the plant so that the dominant poles of the closed-loop system are located at $s = -2 \pm j2$.

(b) Obtain the steady state error to step, ramp, and parabolic inputs for the closed-loop system.

(a) See next page

(b)

Error terms

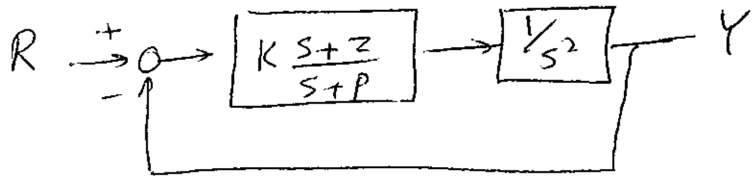
Step $\longrightarrow$ $e_{ss} = 0$

Ramp $\longrightarrow$ $e_{ss} = 0$

Parabalic $\longrightarrow$ $E = \lim_{s \to 0} \dfrac{s \cdot \frac{1}{s^3}}{1 + K\frac{s+z}{s+p}\frac{1}{s^2}}$

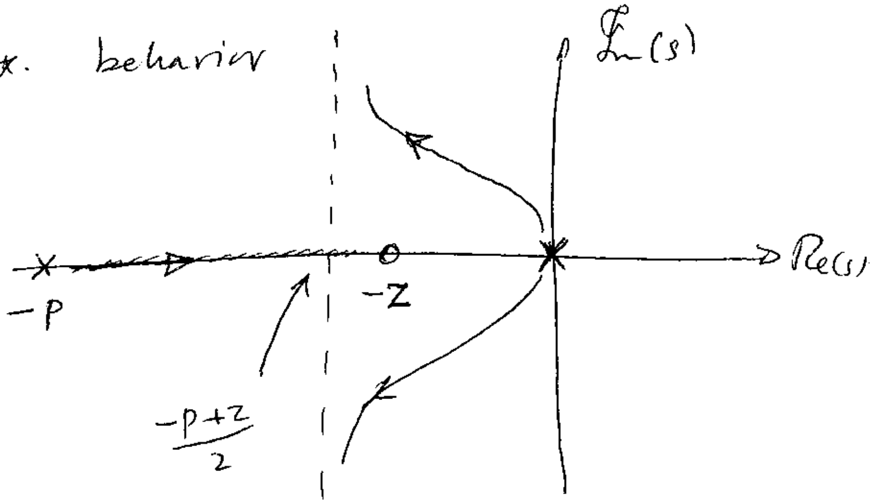
$= \frac{1}{\ }= \frac{P}{\ } = \frac{70}{\ }$

5.26



Dominant poles → $s = -2 \pm j2$

* Root locus shows approx. behavior
* lead compensator → $z < p$



* Asymptotes

$$\frac{\sum \text{poles} - \sum \text{zeros}}{3-1} = \frac{-p+z}{2}$$

angles → $\dfrac{180+360l}{3-1} = \pm 90^\circ$

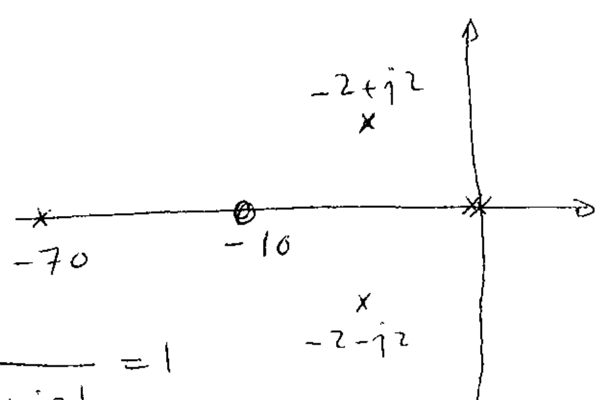
To have dominant poles at $-2 \pm j2$ → $\dfrac{z-p}{2} \ll -2 \rightarrow z-p \ll -4$

$$\boxed{p \gg 4+z}$$

For pole to be dominant we also set to $z \gg 2$

Take
$$\begin{cases} p = 5\,(4+z) \longrightarrow p = 5\,(4+10) = 70 \\ z = 5 \times 2 = 10 \end{cases}$$

To find K:
use magnitude condition at $-2+j2$



$$K \left| \frac{(s+z)}{s^2(s+p)} \right| = 1 \rightsquigarrow K \frac{|-2+j2+10|}{|2+j2|^2\,|70-2+j2|} = 1$$

$$K \frac{\sqrt{8^2+2^2}}{8 \times \sqrt{68^2+2^2}} = 1 \longrightarrow \boxed{K = 65.9} \simeq 66 \Rightarrow \text{Controller } 66\,\frac{s+10}{s+70}$$