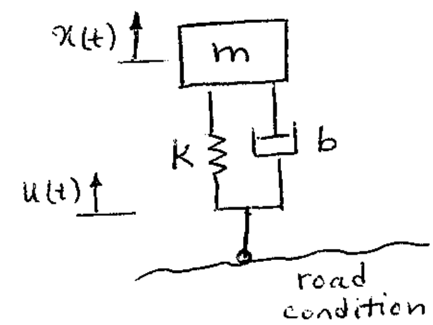
Midterm Exam: Feedback Control Systems (ENSC383), Fall 2007

School of Engineering Science, Simon Fraser University

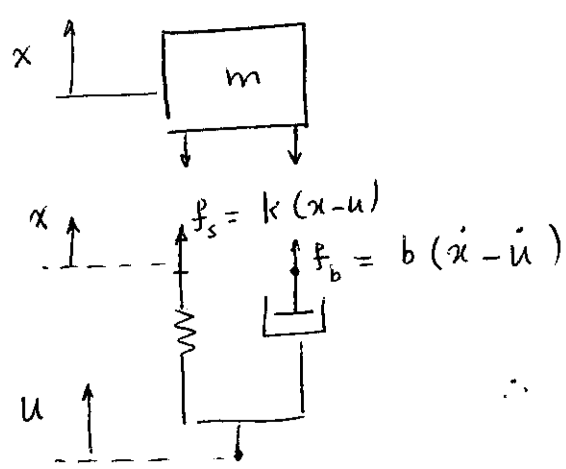
Note: Questions are marked out of 22. This exam has 22% of the overall mark

Name: **Time: 1hr:30min**

1. **(5 marks)** A simplified car model is given by the figure below, where $\boldsymbol{b}$ is the damping coefficient, $\boldsymbol{m}$ is the mass of the car, and $\boldsymbol{k}$ is the stiffness constant of the springs. Write the dynamic equations of the system and obtain the transfer function from input $\boldsymbol{u}$ to output $\boldsymbol{x}$.



Free body diagram



$$\Rightarrow \quad -k(x-u) - b(\dot{x}-\dot{u}) = m\ddot{x}$$

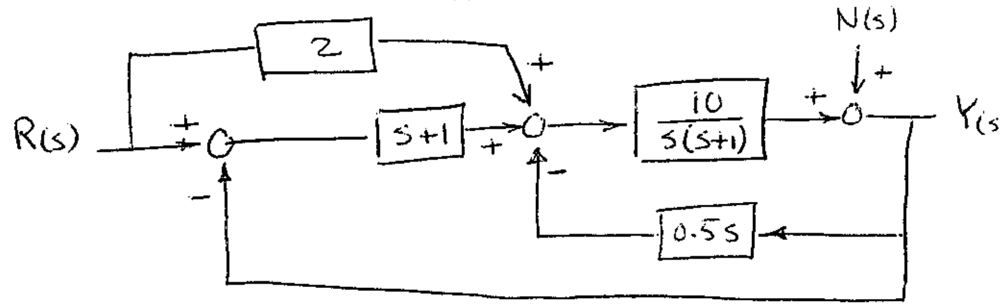
$$\therefore \quad m\ddot{x} + b\dot{x} + kx = b\dot{u} + ku$$

Laplace domain:

$$(ms^2 + bs + k)\,X(s) = (bs+k)\,U(s)$$

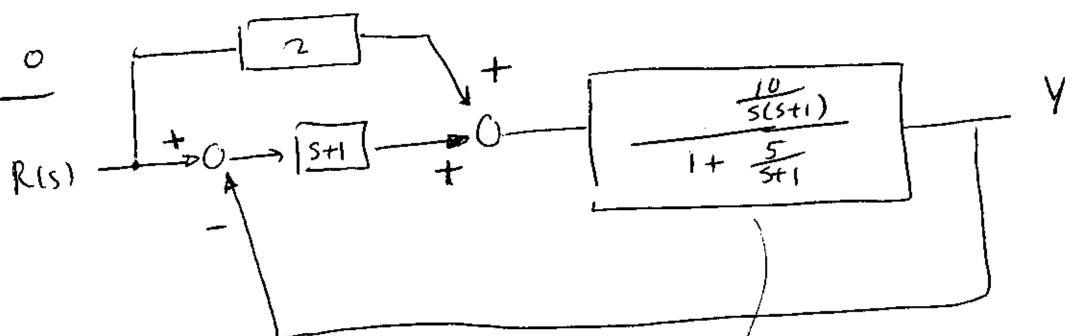
$$\frac{X(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}$$

2. **(5 marks)** The block diagram of a feedback control system is shown in the following figure. Obtain the output Y(s) in terms of R(s) and N(s).



Let us use superposition $\longrightarrow$ $Y(s) = H_R(s)\, R(s) + H_N(s)\, N(s)$

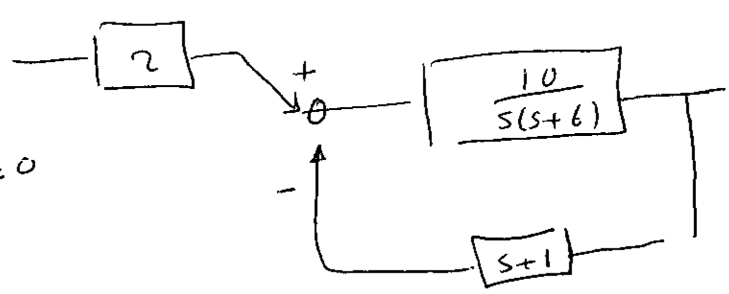
$N(s) = 0$



$$\frac{10}{s^2+s+5s} = \frac{10}{s(s+6)}$$

$R_2 = R$

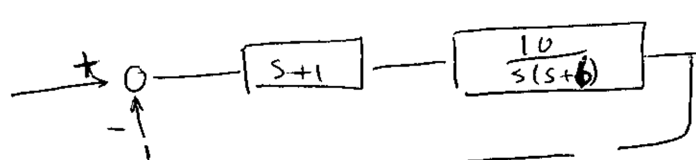
$R_1 = 0$



$$Y_{R_2=R} = 2\,\frac{\frac{10}{s(s+6)}}{1+\frac{10(s+1)}{s(s+6)}} = \frac{20}{s^2+6s+10s+10} = \frac{20}{s^2+16s+10}$$

$R_2 = 0$

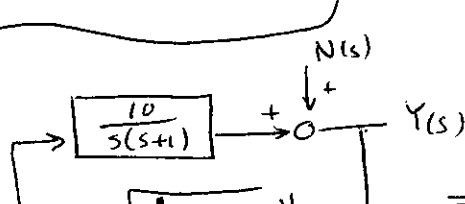
$R_1 = R$



$$Y_{R_1=R} = \frac{\frac{10(s+1)}{s(s+6)}}{1+\frac{10(s+1)}{s(s+6)}} = \frac{10(s+1)}{s^2+6s+10s+10} = \frac{10(s+1)}{s^2+16s+10}$$

$$\therefore\ H_R(s) = \frac{10s+30}{s^2+16s+10}$$

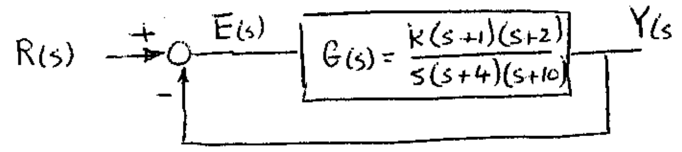
$R = 0$



$$\Longrightarrow\ \frac{Y(s)}{N(s)} = \frac{1}{1+\frac{10(0.5s+1)}{s(s+1)}} = \frac{s^2+s}{s^2+16s+10}$$

Thus $Y(s) = \frac{10s+30}{s^2+16s+10}\,R(s) + \frac{s(s+1)}{s^2+16s+10}\,N(s)$

3. Consider the unity feedback system shown below.
   a. **(4 marks)** Find the range of k for which the closed-loop system is stable
   b. **(3 marks)** What is the steady-state error for unit ramp and step inputs.



(a) Closed loop system:

$$\frac{Y(s)}{R(s)} = \frac{\frac{k(s+1)(s+2)}{s(s+4)(s+10)}}{1 + \frac{k(s+1)(s+2)}{s(s+4)(s+10)}} = \frac{k(s+1)(s+2)}{s(s^2+14s+40) + k(s^2+3s+2)}$$

$$s^3 + 14s^2 + 40s + ks^2 + 3ks + 2k = 0 \quad \leftarrow \text{char. eq.}$$

$$s^3 + (14+k)s^2 + (40+3k)s + 2k = 0$$

Routh's Test

s^3	1	$40+3k$
s^2	$14+k$	$2k$
s^1	$\frac{(14+k)(40+3k)-2k}{14+k}$	
s^0	$2k$	

$14+k > 0 \longrightarrow k > -14$

$2k > 0 \longrightarrow k > 0$

$$\frac{(14+k)(40+3k)-2k}{14+k} > 0 \longrightarrow 14\times 40 + 42k + 40k + 3k^2 - 2k > 0$$

$$3k^2 + 80k + 560 > 0$$

$$k_1, k_2 = \frac{-40 \pm \sqrt{1600 - 3\times 560}}{3}$$

Complex roots

Thus always positive

Hence $\boxed{k > 0}$

(b) ~~Unit step~~ $\Rightarrow$

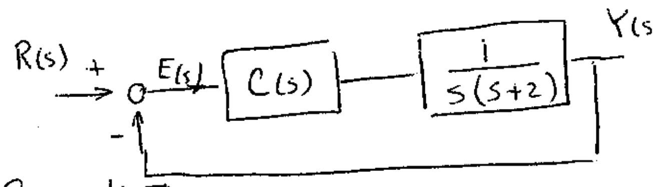
$$\frac{E(s)}{R(s)} = \frac{1}{1 + \frac{k(s+1)(s+2)}{s(s+4)(s+10)}} = \frac{s(s+4)(s+10)}{s(s+4)(s+10) + k(s+1)(s+2)}$$

$$R(s) = \frac{1}{s^2} \Rightarrow E(s) = \frac{s(s+4)(s+10)}{s\left[s(s+4)(s+10) + k(s+1)(s+2)\right]}$$

$$e(t) = \lim_{s\to 0} sE(s) = \frac{4\times 10}{k\times 2} = \frac{20}{k}$$

$$R(s) = \frac{1}{s} \Rightarrow E(s) = 0$$

4. **(5 marks)** In the feedback system below, the desired system response to a step input is specified as peak time t_p = *1 sec* and overshoot M_p = *5%.* Determine whether both specifications can be met simultaneously by selecting *C(s)* as a proportional controller and a proportional-derivative controller.
obtain the corresponding controllers.



C(s) : Proportional controller, i.e., $C(s) = K$

$$\therefore \frac{Y}{R} = \frac{\frac{K}{s(s+2)}}{1+\frac{K}{s(s+2)}} = \frac{K}{s^2+2s+K}$$

$$\left.\begin{array}{l} \omega_n^2 = K \\ 2\zeta\omega_n = 2 \end{array}\right. \Longrightarrow \begin{array}{l} \omega_n = \sqrt{K} \\ 2\zeta\sqrt{K} = 2 \Rightarrow \zeta = \frac{1}{\sqrt{K}} \end{array} \Bigg\} \Rightarrow \begin{array}{c} K = \omega_n^2 \\ \Uparrow \\ \omega_n = \sqrt{K} \\ \zeta = \frac{1}{\sqrt{K}} \\ \Downarrow \\ K = \frac{1}{\zeta^2} \end{array}$$

$$\left\{\begin{array}{l} t_p = 1 = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \Longrightarrow \omega_n = \frac{\pi}{\sqrt{1-\zeta^2}} \\ M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.05 \longrightarrow \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(0.05) \end{array}\right.$$

$$\frac{\zeta^2\pi^2}{1-\zeta^2} = (\ln(0.05))^2 = 8.9744$$

$$\zeta^2 = 0.9093(1-\zeta^2)$$

$$(1+0.9093)\zeta^2 = 0.9093$$

$\zeta = 0.6899$

$$(*)\ \boxed{\zeta^2 = 0.476} \Rightarrow (**)\ \boxed{\omega_n = \frac{\pi}{\sqrt{1-\zeta^2}} = 4.3399}$$

$$K = 2.1 \qquad\qquad K = 18.83$$

$$C(s) = K_p + K_d s \Longrightarrow \frac{Y}{R} = \frac{\frac{(K_p+K_d s)}{s(s+2)}}{1+\frac{K_p+K_d s}{s(s+2)}} = \frac{K_p+K_d s}{s^2+(2+K_d)s+K_p}$$

$$(*)\ \&\ (**) \Longrightarrow \boxed{K_p = \omega_n^2 = 18.83}$$

$$2+K_d = 2\zeta\omega_n \Longrightarrow 2+K_d = 2\times 0.6899\times 4.3399$$

$$\boxed{K_d = 3.9884} \Longrightarrow$$ Can meet spec's

Check zero & poles : $s^2 + 5.9884s + 18.83 = 0$

$$s_1, s_2 = \frac{-5.9884 \pm \sqrt{5.9884^2 - 4\times 18.83}}{2\times 1} = -2.994 \pm j3.14$$

zero $\longrightarrow$ $-\frac{K_p}{K_d} = \frac{-18.83}{3.9884} = -4.7$