SIMON FRASER UNIVERSITY
School of Engineering Science
ENSC 380 Linear Systems

REFRESHER ON COMPLEX NUMBERS AND VARIABLES

1. REPRESENTATION AND CONVERSION

Complex numbers contain the square root of -1. There are two common formats for representing them: Cartesian (rectangular) and polar. If z is a complex variable, they are

Cartesian	Polar
$z = x + j \cdot y$	$z = r \cdot e^{j \cdot \theta}$

where

$x = \mathrm{Re}(z)$	real part	$r = \lvert z \rvert$	magnitude (radius, length)
$y = \mathrm{Im}(z)$	imaginary part	$\theta = \arg(z)$	phase

You can convert between representations:

Cartesian from polar

$x = r \cdot \cos(\theta)$ $\quad$ $y = r \cdot \sin(\theta)$ $\quad$ (in particular, $e^{j\theta} = \cos\theta + j \sin\theta$ - see Appendix)

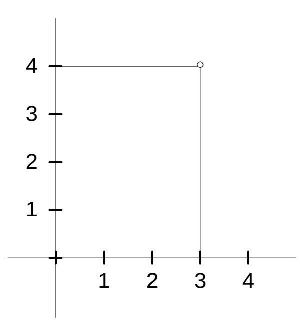
Polar from Cartesian

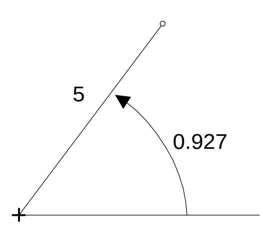
$r = \sqrt{x^2 + y^2}$ $\quad$ $\theta = \mathrm{atan}\left(\frac{y}{x}\right)$

Be cautious with this one, since it reduces all angles to $(-\pi/2, \pi/2)$, and 2nd and 3rd quadrants are lost. To do it right, if $x<0$, then add π to the phase (or subtract π). Draw a picture if you're confused.

Example:

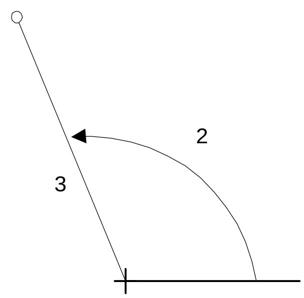
$z := 3 + j \cdot 4$ $\qquad$ $\lvert z \rvert = 5$ $\qquad$ $\arg(z) = 0.927$ radian

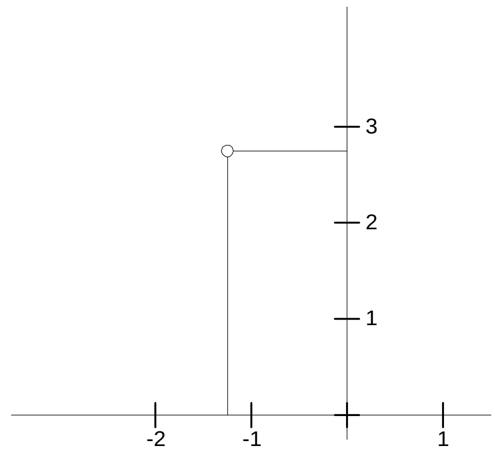




Example:

$z := 3 \cdot e^{j \cdot 2}$ $\qquad$ $\mathrm{Re}(z) = -1.248$ $\qquad$ $\mathrm{Im}(z) = 2.728$





2. BASIC ARITHMETIC OPERATIONS

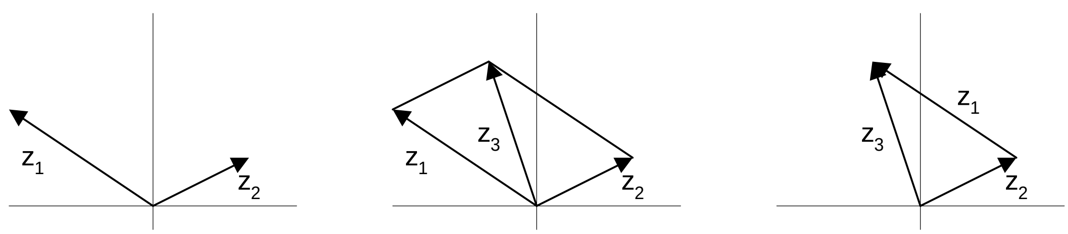
2.1 Addition

Addition is most easily performed in rectangular coordinates, since

$z_3 = z_1 + z_2$ $\qquad$ implies $\qquad$ $\mathrm{Re}(z_3) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$

$$\mathrm{Im}(z_3) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$$

This is just like adding vectors, and it can be visualized in the same way:



parallelogram $\qquad$ head to tail

Example: Add z_1 and z_2

$z_1 := 2 + j \cdot 3$ $\qquad$ $z_2 := -4 + j$ $\qquad$ $z_1 + z_2 = -2 + 4j$

Example: Add z_1 and z_2

$z_1 := 2 \cdot e^{j \cdot 3}$ $\qquad$ $z_2 := 3 \cdot e^{-j \cdot 2}$

Convert to cartesian and add:

$z_1 = -1.98 + 0.282j \qquad z_2 = -1.248 - 2.728j \qquad z_1 + z_2 = -3.228 - 2.446j$

If you want to convert the sum back to polars, you have

$\left| z_1 + z_2 \right| = 4.05 \qquad \arg\left(z_1 + z_2\right) = -2.493$

2.2 Multiplication and Division

Multiplication and division are easily performed in polar coordinates, since

$z_1 = r_1 \cdot e^{j \cdot \theta_1} \qquad z_2 = r_2 \cdot e^{j \cdot \theta_2}$ implies $z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{j \cdot \left(\theta_1 + \theta_2\right)}$

That is,

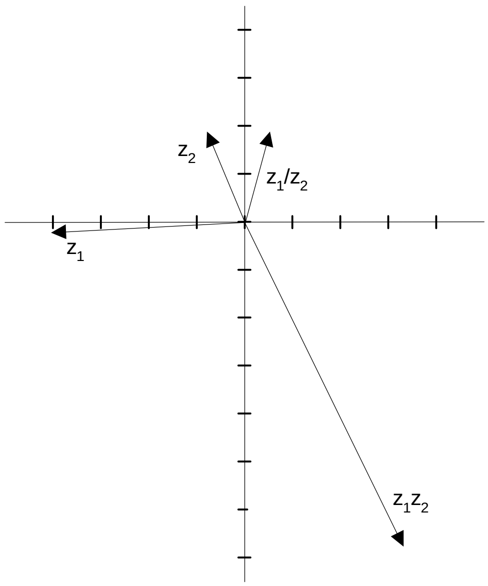
$\left| z_1 \cdot z_2 \right| = \left| z_1 \right| \cdot \left| z_2 \right|$ and $\arg\left(z_1 \cdot z_2\right) = \arg\left(z_1\right) + \arg\left(z_2\right)$

where the phase summation is usually interpreted modulo 2π. The same logic gives, for division,

$\left| \frac{z_1}{z_2} \right| = \frac{\left| z_1 \right|}{\left| z_2 \right|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg\left(z_1\right) - \arg\left(z_2\right)$

Example: Obtain product and quotient in polars

$z_1 = 4 \cdot e^{-j \cdot 3} \qquad z_2 = 2 \cdot e^{j \cdot 2} \qquad z_1 \cdot z_2 = 8 \cdot e^{-j} \qquad \frac{z_1}{z_2} = 2 \cdot e^{-j \cdot 5}$



Example: Obtain product $z_1 z_2$ and quotient z_1/z_2 in polars

$z_1 = 2 + j \cdot 3$ $\qquad\qquad$ $z_2 = 1 - j \cdot 2$

Convert to polars:

$z_1 = 3.6 \cdot e^{j \cdot 0.98}$ $\qquad\qquad$ $z_2 = 2.2 \cdot e^{-j \cdot 1.1}$

Product and quotient in polars (and cartesian, if desired):

$z_1 \cdot z_2 = 8.1 \cdot e^{-j \cdot 0.12}$ $\qquad\qquad$ $\dfrac{z_1}{z_2} = 0.62 \cdot e^{-j \cdot 2.09}$

$\blacksquare = 8 - j$ $\qquad\qquad$ $\blacksquare = -0.31 - j \cdot 0.54$

You can perform multiplication just as easily in Cartesian coordinates by explicit term-by-term multiplication

$$(x_1 + j \cdot y_1) \cdot (x_2 + j \cdot y_2) = x1 \cdot x2 + j^2 \cdot y_1 \cdot y_2 + j \cdot x_1 \cdot y_2 + j \cdot y_1 \cdot x_2$$

$$\blacksquare = x_1 \cdot x_2 - y_1 \cdot y_2 + j \cdot (x_1 \cdot y_2 + y_1 \cdot x_2)$$

As for division in Cartesian coordinates, it's a bit trickier, so we'll revisit it after looking at complex conjugates.

3. COMPLEX CONJUGATES

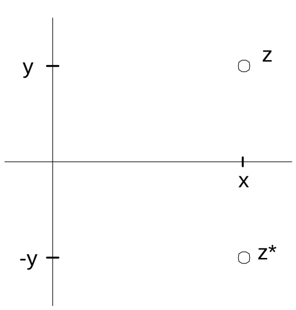
The conjugate of a number has the sign of its imaginary part reversed. It is usually denoted by an asterisk, e.g., z*, but Mathcad uses the less common notation of an overhead bar. Here's an example:

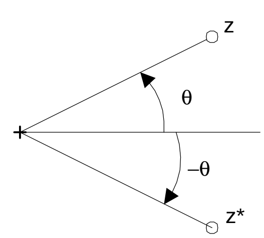
$z = x + j \cdot y$ $\qquad\qquad$ $\bar{z} = x - j \cdot y$

Equivalently, the conjugate of a number has the sign of its phase reversed (why is it equivalent?):

$z = r \cdot e^{j \cdot \theta}$ $\qquad\qquad$ $\bar{z} = r \cdot e^{-j \cdot \theta}$

The drawing shows that complex conjugates are reflections through the real axis.





Complex conjugate notation provides some very useful operations. For starters, see what you get when you multiply a number by its conjugate. If z=x+jy, :

$$z \cdot \overline{z} = (x + j \cdot y) \cdot (x - j \cdot y) = x^2 + y^2 \qquad \text{rectangular}$$

$$z \cdot \overline{z} = r \cdot e^{j \cdot \theta} \cdot r \cdot e^{-j \cdot \theta} = r^2 \qquad \text{polar}$$

Interesting - the product of a number and its conjugate is the sum of squares of the components, or the squared radius, or squared magnitude $|z|^2$. If you think of the number as a vector, then you have just obtained the dot (inner) product of the vector with itself.

More generally, you can multiply a number z_1 by the conjugate of some other number z_2. This, too, has an interesting interpretation. It's easiest to see it in polars:

$$z_1 \cdot \overline{z_2} = r_1 \cdot r_2 \cdot e^{j \cdot (\theta_1 - \theta_2)}$$

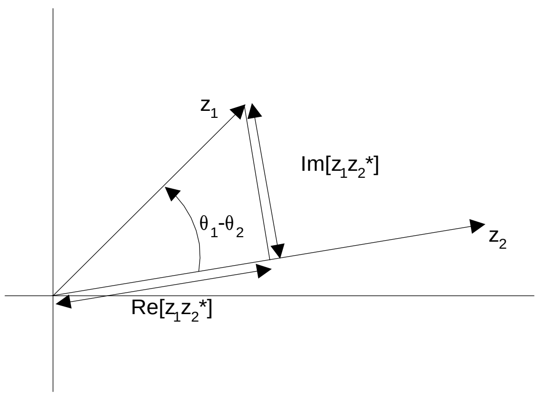
The magnitude of the product is the product of the magnitudes, just as in the product $z_1 z_2$, but the phase is that of z_1 "derotated" by the phase of z_2. That's useful in itself, but now take the real part:

$$\mathrm{Re}\left(z_1 \cdot \overline{z_2}\right) = \mathrm{Re}\left[r_1 \cdot r_2 \cdot e^{j \cdot (\theta_1 - \theta_2)}\right] = r_1 \cdot r_2 \cdot \cos(\theta_1 - \theta_2)$$

This is the dot product of z_1 and z_2 if we think of them as vectors - a nice geometric interpretation. In fact, you can interpret

$$\mathrm{Re}\left(z_1 \cdot \frac{\overline{z_2}}{|z_2|}\right) \qquad \text{and} \qquad \mathrm{Im}\left(z_1 \cdot \frac{\overline{z_2}}{|z_2|}\right)$$

as the components of z_1 in the directions parallel and perpendicular to z_2, as shown in the sketch below.



Here are two more interpretations. The average value of the quantity $\mathrm{Re}[z_1 z_2^*]$ is commonly used in signal processing and communications as the correlation between two complex signals. Moving to electric circuits, if AC current and voltage are represented by the phasors (complex numbers) I and V, then 0.5 Re[IV*] is the average power (averaged over one cycle).

Now, back to complex division in rectangular coordinates. If

$$z_1 = x_1 + j\cdot y_1 \qquad\qquad z_2 = x_2 + j\cdot y_2$$

then

$$\frac{z_1}{z_2} = \frac{x_1 + j\cdot y_1}{x_2 + j\cdot y_2}$$

This looks ugly. But just multiply numerator and denominator by the conjugate of z_2:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\cdot\frac{\overline{z_2}}{\overline{z_2}} = \frac{(x_1 + j\cdot y_1)}{(x_2 + j\cdot y_2)}\cdot\frac{(x_2 - j\cdot y_2)}{(x_2 - j\cdot y_2)} = \frac{x_1\cdot x_2 + y_1\cdot y_2 + j\cdot(y_1\cdot x_2 - x_1\cdot y_2)}{{x_2}^2 + {y_2}^2}$$

This is a straightforward calculation. It looks somewhat laborious, but would you really prefer to convert both numbers to polars, then convert the quotient back to Cartesian?

APPENDIX: THE COMPLEX EXPONENTIAL

You've seen that exp(jθ)=cos(θ)+jsin(θ). It's called Euler's identity, and it will be part of your mental landscape from now on. But why is it true? Here you'll see two reasons.

The first argument uses the series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \blacksquare + \blacksquare + \blacksquare$$

Substitute x=jθ and collect real and imaginary terms:

$$e^{j\cdot\theta} = 1 + j\cdot\theta - \frac{\theta^2}{2!} - j\cdot\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \blacksquare + \blacksquare + \blacksquare$$

$$\blacksquare = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \blacksquare + \blacksquare + j\cdot\left(\theta - \frac{\theta^3}{3!} + \blacksquare + \blacksquare\right)$$

$$\blacksquare = \cos(\theta) + j\cdot\sin(\theta)$$

Now for the second demonstration. A defining property of exponentials is that they reproduce themselves under differentiation. The unique solution of the elementary differential equation

$$\frac{d}{d\theta} f(\theta) = j \cdot f(\theta) \quad \text{with initial condition } f(0) = 1 \quad \text{is} \quad f(\theta) = e^{j \cdot \theta}$$

However, f(θ)=cos(θ)+jsin(θ) also satisfies the equation, since

$$\frac{d}{d\theta}(\cos(\theta) + j \cdot \sin(\theta)) = -\sin(\theta) + j \cdot \cos(\theta) = j \cdot (\cos(\theta) + j \cdot \sin(\theta))$$

Therefore $e^{j \cdot \theta} = \cos(\theta) + j \cdot \sin(\theta)$

QUESTIONS

1. Convert to the other representation (polar or cartesian/rectangular)

$3+j\cdot 5$ $\quad$ $-2+j\cdot 3$ $\quad$ $x+j\cdot 4$ $\quad$ $\dfrac{1}{\sqrt{2}}+\dfrac{j}{\sqrt{2}}$

$2\cdot e^{-j\cdot 3}$ $\quad$ $2\cdot e^{j\cdot\theta}$ $\quad$ $r\cdot e^{j\cdot 2}$

$2\cdot e^{2+j\cdot 3}$ (sum of exponents implies a product)

e^{0} $\quad$ $e^{j\cdot\frac{\pi}{2}}$ $\quad$ $e^{j\cdot\pi}$ $\quad$ $e^{j\cdot\frac{3}{2}\cdot\pi}$ $\quad$ $e^{-j\cdot\frac{\pi}{2}}$

2. Perform the indicated sums and obtain the results in both representations

$3\cdot e^{j}-4\cdot e^{-j\cdot 2}$ $\quad$ $(p+j\cdot 3)+\left(2\cdot e^{-j\cdot 7}\right)$ $\quad$ $(z+j\cdot 6)+(3-j\cdot 3)$

3. Perform the indicated multiplications or divisions and obtain the results in polars

$3\cdot e^{j\cdot 3}\cdot 5\cdot e^{-j\cdot 2}$ $\quad$ $(2-j\cdot 4)\cdot e^{j\cdot 5}$ $\quad$ $\dfrac{2}{3+j\cdot\theta}$ $\quad$ $\dfrac{5\cdot e^{j\cdot\frac{\pi}{2}}}{2j}$

4. Perform the indicated multiplications or divisions and obtain the results in Cartesians

$\dfrac{3+j\cdot a}{3-j\cdot a}$ $\quad$ $\dfrac{5\cdot e^{j\cdot\frac{\pi}{2}}}{2j}$